

IRREDUCIBLE COMPLETELY POINTED MODULES OF QUANTUM GROUPS OF TYPE A

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ABSTRACT. We give a classification of all irreducible completely pointed $U_q(\mathfrak{sl}_{n+1})$ modules over a characteristic zero field in which q is not a root of unity. This generalizes the classification result of Benkart, Britten and Lemire in the non quantum case. We also show that any infinite-dimensional irreducible completely pointed $U_q(\mathfrak{sl}_{n+1})$ can be obtained from some irreducible completely pointed module over the quantized Weyl algebra A_{n+1}^q .

Keywords: quantum groups, representation theory, weight modules of bounded multiplicity.

INTRODUCTION

Let $U_q(\mathfrak{g})$ be the quantum group of finite-dimensional semisimple Lie algebra over a characteristic zero field in which q is not a root of unity. A $U_q(\mathfrak{g})$ weight module is called completely pointed if all of its weight spaces are one dimensional. This paper is a generalization of the classification given by Benkart, Britten and Lemire ([1]) of infinite dimensional completely pointed modules of semisimple Lie algebras. In the Lie algebra case such modules can only exist if every ideal of \mathfrak{g} is of type A or C . In the current paper we consider the case of $U_q(\mathfrak{sl}_{n+1})$, i.e. the quantum group of type A .

Throughout the paper, we will make extensive use of certain generators E_α of $U_q(\mathfrak{g})$ introduced by Lusztig ([10]) where α ranges over the roots of \mathfrak{g} , which are analogues of the root vectors of \mathfrak{g} . These generators are not unique, but depend on a choice of reduced decomposition of the longest Weyl group element, w_0 . For a fixed reduced decomposition of w_0 , and a irreducible weight module V we see that each E_α acts either locally nilpotently or injectively on V . This set is very much like root vectors parabolic subalgebra \mathfrak{p} of \mathfrak{sl}_{n+1} though $\mathfrak{sl}_{n+1} \not\subset U_q(\mathfrak{sl}_{n+1})$ as a Lie algebra so this correspondence is not precise. Nevertheless, we call the $U_q(\mathfrak{sl}_{n+1})$ -subalgebra generated by locally nilpotent root vectors $U_q(\mathfrak{p})$ and there exists another $U_q(\mathfrak{sl}_{n+1})$ -submodule $U_q(\mathfrak{u})$ where \mathfrak{u} is analogous to the nilradical of

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p. The first main theorem of the paper is as follows, where V^+ is the \mathfrak{u} -invariant subset of V :

Theorem I. *Let V be an irreducible, infinite-dimensional, completely-pointed $U_q(\mathfrak{sl}_{n+1})$ -module and let $v^+ \in V^+$ be given. Then the action of $U_q(\mathfrak{sl}_{n+1})$ on V can be extended to a $U_q(\mathfrak{gl}_{n+1})$ action such that the following relations hold:*

$$E_{-\varepsilon_i+\varepsilon_j} E_{\varepsilon_i-\varepsilon_j} \cdot v^+ = [\bar{K}_i; 1][\bar{K}_j; 0] \cdot v^+.$$

Also, we have

$$F_i E_i \cdot v_\lambda = [\bar{K}_i; 1][\bar{K}_{i+1}; 0] \cdot v_\lambda$$

for any weight vector $v_\lambda \in V$.

Using this theorem one sees, for example, that the action of the cyclic subalgebra $C(U_q(\mathfrak{sl}_{n+1}))$ is completely determined by the action of the \bar{K}_i (see Lemma 3.3), hence this gives a classification of irreducible completely pointed $U_q(\mathfrak{sl}_{n+1})$ modules (with all finite dimensional ones given in Proposition 2.2).

In our next two main results, we construct the infinite-dimensional completely pointed $U_q(\mathfrak{sl}_{n+1})$ -modules. Let A_{n+1}^q be the rank $n+1$ quantum Weyl algebra and π be the homomorphism from $U_q(\mathfrak{gl}_{n+1})$ to A_{n+1}^q (see [6]) which restricts to $U_q(\mathfrak{sl}_{n+1})$. Then we have the following:

Theorem II A. *Let W be an irreducible completely pointed A_{n+1}^q -module. Let π^*W be the $U_q(\mathfrak{gl}_{n+1})$ -module, given as the π -pullback of W . Then π^*W is completely reducible, and each irreducible submodule is completely pointed, and occurs with multiplicity one.*

This gives a construction of irreducible infinite dimensional, completely pointed $U_q(\mathfrak{sl}_{n+1})$ -modules. An application of Theorem I then gives the following, which completes our classification:

Theorem II B. *Any infinite-dimensional irreducible completely pointed $U_q(\mathfrak{sl}_{n+1})$ is isomorphic to a direct summand of π^*W for some irreducible completely pointed A_{n+1}^q -module W .*

1. PRELIMINARIES

Let \mathbb{F} be a field of characteristic 0 closed under quadratic extensions and suppose $q \in \mathbb{F}$ is nonzero and not a root of unity. For us, $U_q(\mathfrak{gl}_{n+1})$ is the associative unital \mathbb{F} -algebra with generators $E_i, F_i, \bar{K}_j^{\pm 1}$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n+1\}$ and defining relations

$$(1) \quad \bar{K}_j E_i \bar{K}_j^{-1} = q^{\delta_{ij}-\delta_{j,i+1}} E_i, \bar{K}_j F_i \bar{K}_j^{-1} = q^{-(\delta_{ij}-\delta_{j,i+1})} F_i, i \in \{1, \dots, n\}, j \in \{1, \dots, n+1\}$$

$$(2) \quad [E_i, F_j] = \delta_{ij} \frac{\bar{K}_i \bar{K}_{i+1}^{-1} - \bar{K}_i^{-1} \bar{K}_{i+1}}{q - q^{-1}}, i, j \in \{1, \dots, n+1\}$$

$$(3) [E_i^\pm, E_j^\pm] = 0 \quad \text{for } |i - j| > 1,$$

$$(4) (E_i^\pm)^2 E_j^\pm - [2]_q E_j^\pm E_i^\pm E_j^\pm + E_j^\pm (E_i^\pm)^2 = 0, \quad \text{for } |i - j| = 1,$$

where $E_i^+ := E_i$, $E_i^- := F_i$, and $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$, $k \in \mathbb{Z}_{\geq 0}$. Then, $U_q(\mathfrak{sl}_{n+1})$ is the subalgebra of $U_q(\mathfrak{gl}_{n+1})$ generated by E_i, F_i , and $K_i := \bar{K}_i \bar{K}_{i+1}^{-1}$.

For $\mathfrak{g} = \mathfrak{sl}_{n+1}$ recall the following automorphisms $T_i : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$, $1 \leq i \leq n$ as given by Lusztig [10]:

$$\begin{aligned} T_i(E_j) &= \begin{cases} -F_i K_i, & \text{if } i = j \\ q^{-1} E_j E_i - E_i E_j, & \text{if } |i - j| = 1 \\ E_j, & \text{otherwise,} \end{cases} \\ T_i(F_j) &= \begin{cases} -K_i^{-1} E_i, & \text{if } i = j \\ -F_j F_i + q F_i F_j, & \text{if } |i - j| = 1 \\ F_j, & \text{otherwise,} \end{cases} \\ T_i(K_j) &= \begin{cases} K_j^{-1}, & \text{if } i = j, \\ K_i K_j, & \text{if } |i - j| = 1, \\ K_j, & \text{otherwise.} \end{cases} \end{aligned}$$

We also recall the braid relations satisfied by the T_i :

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \\ T_i T_j &= T_j T_i, \text{ if } |i - j| > 1. \end{aligned}$$

To each root α we assign a corresponding root vector E_α in using following method. Let $w_0 = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced decomposition of the longest Weyl group element. Then every positive root occurs exactly once in the following sequence:

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_r = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}(\alpha_{i_r}).$$

The positive root vector E_{β_k} is defined to be $T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(E_{i_k})$, and the negative root vector $E_{-\beta_k}$ is defined by the same sequence of T_i 's acting on F_{i_k} . We choose $w_0 = s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-1} \cdots s_1 s_2 s_1$ as our reduced expression of the longest Weyl group element of $U_q(\mathfrak{sl}_{n+1})$, which gives the following sequence of positive roots:

$$\begin{aligned} \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_4, \dots, \varepsilon_1 - \varepsilon_{n+1}, \\ \varepsilon_2 - \varepsilon_3, \varepsilon_2 - \varepsilon_4, \dots, \varepsilon_2 - \varepsilon_{n+1}, \\ \dots \\ \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n - \varepsilon_{n+1}, \\ \varepsilon_n - \varepsilon_{n+1}. \end{aligned}$$

We recall also from [3, Example 8.1.5] the following identity:

$$T_i T_{i+1}(E_i) = E_{i+1}.$$

Using this fact, and the braid relations, one obtains the following simplified form for the root vectors:

$$\begin{aligned} E_{\varepsilon_1-\varepsilon_2} &= E_1, E_{\varepsilon_1-\varepsilon_3} = T_1(E_2), E_{\varepsilon_1-\varepsilon_4} = T_1T_2(E_3), \dots, E_{\varepsilon_1-\varepsilon_{n-1}} = T_1T_2 \cdots T_{n-1}(E_n), \\ E_{\varepsilon_2-\varepsilon_3} &= E_2, E_{\varepsilon_2-\varepsilon_4} = T_2(E_3), \dots, E_{\varepsilon_2-\varepsilon_{n+1}} = T_2T_3 \cdots T_{n-1}(E_n), \\ &\dots \\ E_{\varepsilon_{n-1}-\varepsilon_n} &= E_{n-1}, E_{\varepsilon_{n-1}-\varepsilon_{n+1}} = T_{n-1}(E_n), \\ E_{\varepsilon_n-\varepsilon_{n+1}} &= E_n \end{aligned}$$

and similarly for the negative root vectors. Let $1 \leq i < j < k \leq n$. Double induction on i and j gives the following, where $[x, y]_v = xy - yx$:

$$E_{\varepsilon_i-\varepsilon_k} = -[E_{\varepsilon_i-\varepsilon_j}, E_{\varepsilon_j-\varepsilon_k}]_{q^{-1}}, \quad (1.1)$$

$$E_{-\varepsilon_i+\varepsilon_k} = -[E_{-\varepsilon_j+\varepsilon_k}, E_{-\varepsilon_i+\varepsilon_j}]_q. \quad (1.2)$$

Similarly we have:

$$[E_{\varepsilon_j-\varepsilon_k}, E_{-\varepsilon_i+\varepsilon_k}] = -qK_{jk}^{-1}E_{-\varepsilon_i+\varepsilon_j}, \quad [E_{\varepsilon_i-\varepsilon_k}, E_{-\varepsilon_j+\varepsilon_k}] = -K_{jk}E_{\varepsilon_i-\varepsilon_j}, \quad (1.3)$$

$$[E_{\varepsilon_i-\varepsilon_j}, E_{-\varepsilon_i+\varepsilon_k}] = K_{ij}E_{-\varepsilon_j+\varepsilon_k}, \quad [E_{\varepsilon_i-\varepsilon_k}, E_{-\varepsilon_i+\varepsilon_j}] = q^{-1}K_{ij}^{-1}E_{\varepsilon_j-\varepsilon_k}, \quad (1.4)$$

where $K_{ij} = \prod_{k=i}^{j-1} K_k$. Also, relations (3) and (4) of the definition of $U_q(\mathfrak{sl}_{n+1})$ lead to the following:

$$E_{\varepsilon_j-\varepsilon_k}E_{\varepsilon_i-\varepsilon_k} = q^{-1}E_{\varepsilon_i-\varepsilon_k}E_{\varepsilon_j-\varepsilon_k}, E_{\varepsilon_i-\varepsilon_k}E_{\varepsilon_i-\varepsilon_j} = q^{-1}E_{\varepsilon_i-\varepsilon_j}E_{\varepsilon_i-\varepsilon_k}, \quad (1.5)$$

$$E_{-\varepsilon_j+\varepsilon_k}E_{-\varepsilon_i+\varepsilon_k} = q^{-1}E_{-\varepsilon_i+\varepsilon_k}E_{-\varepsilon_j+\varepsilon_k}, E_{-\varepsilon_i+\varepsilon_k}E_{-\varepsilon_i+\varepsilon_j} = q^{-1}E_{-\varepsilon_i+\varepsilon_j}E_{-\varepsilon_i+\varepsilon_k}. \quad (1.6)$$

Finally, if $1 \leq i < j < k < l \leq n$ then:

$$[E_{\varepsilon_i-\varepsilon_j}, E_{-\varepsilon_k+\varepsilon_l}] = [E_{\varepsilon_i-\varepsilon_j}, E_{\varepsilon_k-\varepsilon_l}] = [E_{\varepsilon_i-\varepsilon_l}, E_{\varepsilon_j-\varepsilon_k}] = 0 \quad (1.7)$$

$$[E_{\varepsilon_i-\varepsilon_k}, E_{\varepsilon_j-\varepsilon_l}] = (q - q^{-1})E_{\varepsilon_i-\varepsilon_l}E_{\varepsilon_j-\varepsilon_k} \quad (1.8)$$

$$[E_{-\varepsilon_i+\varepsilon_k}, E_{-\varepsilon_j+\varepsilon_l}] = (q - q^{-1})E_{-\varepsilon_i+\varepsilon_l}E_{-\varepsilon_j+\varepsilon_k} \quad (1.9)$$

$$[E_{\varepsilon_i-\varepsilon_k}, E_{-\varepsilon_j+\varepsilon_l}] = -(q - q^{-1})K_{jk}E_{-\varepsilon_i+\varepsilon_j}E_{\varepsilon_k-\varepsilon_l} \quad (1.10)$$

$$[E_{\varepsilon_j-\varepsilon_l}, E_{-\varepsilon_i+\varepsilon_k}] = (q - q^{-1})K_{jk}^{-1}E_{\varepsilon_k-\varepsilon_l}E_{-\varepsilon_i+\varepsilon_j}. \quad (1.11)$$

Finally, since the T_i are $U_q(\mathfrak{g})$ automorphisms, we have:

$$[E_{\varepsilon_i-\varepsilon_j}, E_{-\varepsilon_i+\varepsilon_j}] = [K_{ij}; 0] \quad (1.12)$$

where $[K; j] = \frac{q^j K - q^{-j} K^{-1}}{q - q^{-1}}$ for invertible $K \in \mathbb{F}[K_1^{\pm 1}, K_2^{\pm 1}, \dots, K_n^{\pm 1}]$ and $j \in \mathbb{Z}$.

Let V be a $U_q(\mathfrak{sl}_{n+1})$ -module. For $\lambda \in (\mathbb{F}^\times)^n$, the weight space V_λ is defined to be the subspace $\{v \in V \mid K_i \cdot v = \lambda_i v\}$. It is easy to show that the sum of weight spaces in V over all $\lambda \in (\mathbb{F}^\times)^n$ is direct. Moreover, if V is finite-dimensional then it is the sum of its weight spaces (see [3]) though the same is not necessarily true if V is infinite-dimensional. A $U_q(\mathfrak{sl}_{n+1})$ -module that is the direct sum of its weight spaces is called a $U_q(\mathfrak{sl}_{n+1})$ weight module. Throughout this paper, we will consider only irreducible modules in the category of $U_q(\mathfrak{sl}_{n+1})$ weight modules.

2. CLASSIFICATION OF IRREDUCIBLE COMPLETELY POINTED MODULES

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\Phi = \Phi(\mathfrak{g})$ be the root system of \mathfrak{g} . Let V be an irreducible $U_q(\mathfrak{g})$ weight module and $\alpha \in \Phi$. On a weight module, the only possible eigenvalue of E_α is 0, hence E_α either acts nilpotently or injectively on a given weight vector. The subset of vectors on which E_α acts nilpotently (resp. injectively) is a submodule of V . Since V is irreducible, we see that E_α acts nilpotently on all of V or else it acts injectively. In the first case, E_α is called *locally nilpotent* and in the second it is called *torsion free*. Highest weight modules are a special case where every positive root vector E_α is locally nilpotent. The other extreme is where every root vector is torsion free. Finally, there are cases where a certain subset of positive root vectors are locally nilpotent but not necessarily all of them. We discuss each case below.

2.1. Highest weight modules. The irreducible highest weight $U_q(\mathfrak{g})$ -module with highest weight $\lambda \in (\mathbb{F}^\times)^n$ is denoted $L(\lambda)$. Note that for us, unlike the finite-dimensional case for example, K_i can have arbitrary eigenvalues in $\mathbb{F} \setminus \{0\}$, not just powers of q . Also this is done so that we can have examples of torsion-free modules (see below).

Lemma 2.1. *Assume V is a completely pointed $U_q(\mathfrak{g})$ -module and $v \in V$ is a weight vector. For θ in the positive root lattice, suppose $x_1, x_2 \in U_q(\mathfrak{g})_\theta$ and $y_1, y_2 \in U_q(\mathfrak{g})_{-\theta}$. Then $y_i x_j \cdot v = \gamma_{i,j} v$ for some $\gamma_{i,j} \in \mathbb{F}$ and $i, j \in \{1, 2\}$, and the 2×2 matrix (γ_{ij}) is singular.*

Proof. Same as the proof in [1, Lemma 3.2], with \mathbb{F} in place of \mathbb{C} . \square

Proposition 2.2 (Analogous to [1, Proposition 3.2]). *The irreducible highest weight $U_q(\mathfrak{sl}_{n+1})$ -module $L(\lambda)$ is completely pointed only if $\lambda = \pm 1, \lambda_i = \pm q, \lambda_1 = c, \lambda_n = c, \lambda_i \lambda_{i+1} = \pm q^{-1}$ for some $i = 1, 2, \dots, n-1$, where $c \in \mathbb{F}^\times$ is arbitrary and all unspecified entries are ± 1 .*

Proof. Let v^+ be a highest weight vector of $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in (\mathbb{F}^\times)^n$. All irreducible highest weight $U_q(\mathfrak{sl}_2)$ -modules are completely pointed, and equal to $L(c)$ for some c which proves the $n = 1$ case.

Suppose $n > 1$. Let $x_1 = E_{-\varepsilon_i + \varepsilon_{i+2}}, x_2 = E_{-\varepsilon_i + \varepsilon_{i+1}} E_{-\varepsilon_{i+1} + \varepsilon_{i+2}} \in U_q(\mathfrak{g})_{-\varepsilon_i + \varepsilon_{i+2}}$, and $y_1 = E_{\varepsilon_i - \varepsilon_{i+2}}, y_2 = E_{\varepsilon_i - \varepsilon_{i+1}} E_{\varepsilon_{i+1} - \varepsilon_{i+2}} \in U_q(\mathfrak{g})_{\varepsilon_i - \varepsilon_{i+2}}$ for $i \in \{1, 2, \dots, n-1\}$. We denote $\lambda_{ij} = \prod_{k=i}^{j-1} \lambda_k$ and compute:

$$\begin{aligned} y_1 x_1 \cdot v_\lambda &= E_{\varepsilon_i - \varepsilon_{i+2}} E_{-\varepsilon_i + \varepsilon_{i+2}} \cdot v_\lambda \\ &= E_{-\varepsilon_i + \varepsilon_{i+2}} E_{\varepsilon_i - \varepsilon_{i+2}} \cdot v_\lambda + [K_{i,i+2}; 0] \cdot v_\lambda \\ &= [\lambda_{i,i+2}; 0] v_\lambda, \\ y_1 x_2 \cdot v_\lambda &= E_{\varepsilon_i - \varepsilon_{i+2}} E_{-\varepsilon_i + \varepsilon_{i+1}} E_{-\varepsilon_{i+1} + \varepsilon_{i+2}} \cdot v_\lambda \\ &= q^{-1} K_i^{-1} E_{\varepsilon_{i+1} - \varepsilon_{i+2}} E_{-\varepsilon_{i+1} + \varepsilon_{i+2}} \cdot v_\lambda \end{aligned}$$

$$\begin{aligned}
&= q^{-1}K_i^{-1}[K_{i+1}; 0] \cdot v_\lambda \\
&= q^{-1}\lambda_i^{-1}[\lambda_{i+1}; 0]v_\lambda, \\
y_2x_1 \cdot v_\lambda &= E_{\varepsilon_i - \varepsilon_{i+1}}E_{\varepsilon_{i+1} - \varepsilon_{i+2}}E_{-\varepsilon_i + \varepsilon_{i+2}} \cdot v_\lambda \\
&= -E_{\varepsilon_i - \varepsilon_{i+1}}qK_{i+1}^{-1}E_{-\varepsilon_i + \varepsilon_{i+1}} \cdot v_\lambda \\
&= -K_{i+1}^{-1}[K_i; 0] \cdot v_\lambda \\
&= -\lambda_{i+1}^{-1}[\lambda_i; 0]v_\lambda, \\
y_2x_2 \cdot v_\lambda &= E_{\varepsilon_i - \varepsilon_{i+1}}E_{\varepsilon_{i+1} - \varepsilon_{i+2}}E_{-\varepsilon_i + \varepsilon_{i+1}}E_{-\varepsilon_{i+1} + \varepsilon_{i+2}} \cdot v_\lambda \\
&= [\lambda_i; 0][\lambda_{i+1}; 0]v_\lambda.
\end{aligned}$$

Therefore, Lemma 2.1 gives:

$$[\lambda_i; 0][\lambda_{i+1}; 0](\lambda_{i,i+2}; 0) + q^{-1}\lambda_{i,i+2}^{-1} = 0, \quad (2.1)$$

from which we see $\lambda_i = \pm 1, \lambda_{i+1} = \pm 1$, or $\lambda_i\lambda_{i+1} = \pm q^{-1}$. This finishes the proof for $n < 3$, so assume $n \geq 3$.

For $1 \leq i < j < k < l \leq n+1$, let $x_1 = E_{-\varepsilon_i + \varepsilon_l}, x_2 = E_{-\varepsilon_i + \varepsilon_k}E_{-\varepsilon_k + \varepsilon_l} \in U_q(\mathfrak{g})_{-\varepsilon_i + \varepsilon_l}$ and $y_1 = E_{\varepsilon_i - \varepsilon_l}, y_2 = E_{\varepsilon_i - \varepsilon_j}E_{\varepsilon_j + \varepsilon_l} \in U_q(\mathfrak{g})_{\varepsilon_i - \varepsilon_l}$. We compute:

$$\begin{aligned}
y_1x_1 \cdot v_\lambda &= E_{\varepsilon_i - \varepsilon_l}E_{-\varepsilon_i + \varepsilon_l} \cdot v_\lambda \\
&= [\lambda_{il}; 0]v_\lambda, \\
y_1x_2 \cdot v_\lambda &= E_{\varepsilon_i - \varepsilon_l}E_{-\varepsilon_i + \varepsilon_k}E_{-\varepsilon_k + \varepsilon_l} \cdot v_\lambda \\
&= q^{-1}K_{ik}^{-1}E_{\varepsilon_k - \varepsilon_l}E_{-\varepsilon_k + \varepsilon_l} \cdot v_\lambda \\
&= q^{-1}\lambda_{ik}^{-1}[\lambda_{kl}; 0]v_\lambda, \\
y_2x_1 \cdot v_\lambda &= E_{\varepsilon_i - \varepsilon_j}E_{\varepsilon_j - \varepsilon_l}E_{-\varepsilon_i + \varepsilon_l} \cdot v_\lambda \\
&= -E_{\varepsilon_i - \varepsilon_j}qK_{jl}^{-1}E_{-\varepsilon_i + \varepsilon_j} \cdot v_\lambda \\
&= -q^2\lambda_{jl}^{-1}[\lambda_{ij}; 0]v_\lambda, \\
y_2x_2 \cdot v_\lambda &= E_{\varepsilon_i - \varepsilon_j}E_{\varepsilon_j - \varepsilon_l}E_{-\varepsilon_i + \varepsilon_k}E_{-\varepsilon_k + \varepsilon_l} \cdot v_\lambda \\
&= E_{\varepsilon_i - \varepsilon_j}E_{-\varepsilon_i + \varepsilon_k}E_{\varepsilon_j - \varepsilon_l}E_{-\varepsilon_k + \varepsilon_l} \cdot v_\lambda \\
&\quad + E_{\varepsilon_i - \varepsilon_j}(q - q^{-1})K_{jk}^{-1}E_{-\varepsilon_i + \varepsilon_j}E_{\varepsilon_k - \varepsilon_l}E_{-\varepsilon_k + \varepsilon_l} \cdot v_\lambda \\
&= (q^2 - 1)K_{jk}^{-1}E_{\varepsilon_i - \varepsilon_j}E_{-\varepsilon_i + \varepsilon_j}E_{\varepsilon_k - \varepsilon_l}E_{-\varepsilon_k + \varepsilon_l} \cdot v_\lambda \\
&= (q^2 - 1)K_{jk}^{-1}[K_{ij}; 0][K_{kl}; 0] \cdot v_\lambda \\
&= (q^2 - 1)\lambda_{jk}^{-1}[\lambda_{ij}; 0][\lambda_{kl}; 0] \cdot v_\lambda.
\end{aligned}$$

It follows from Lemma 2.1 that

$$((q^2 - 1)\lambda_{jk}^{-1}[\lambda_{il}; 0] + q\lambda_{jl}^{-1}\lambda_{ik}^{-1})[\lambda_{ij}; 0][\lambda_{kl}; 0] = q\lambda_{ij}\lambda_{kl}[\lambda_{ij}; 0][\lambda_{kl}; 0] = 0 \quad (2.2)$$

Therefore $\lambda_{ij} = \pm 1$ or $\lambda_{kl} = \pm 1$, for $1 \leq i < j < k < l \leq n+1$. Let i be minimal such that $\lambda_i \neq \pm 1$. Then we have $\lambda_j = \pm 1$ for all $j > i+1$. Since i was chosen to

be minimal such that $\lambda_i \neq \pm 1$ the other index j such that $\lambda_j \neq 0$ is $j = i + 1$. If $\lambda_{i+1} \neq \pm 1$ the previous paragraph implies that $\lambda_i \lambda_{i+1} = \pm q^{-1}$.

This leaves the case such that only $\lambda_i \neq \pm 1$. Let $\lambda_i = c \in \mathbb{F} \setminus \{0\}$ and suppose $1 < i < n + 1$ (if i is not in that range, then c is not fixed in the statement of the theorem). Let $x_1 = E_{-\varepsilon_{i-1}+\varepsilon_{i+1}} E_{-\varepsilon_i+\varepsilon_{i+2}}, x_2 = E_{-\varepsilon_i+\varepsilon_{i+1}} E_{-\varepsilon_{i-1}+\varepsilon_{i+2}}, y_1 = E_{\varepsilon_{i-1}-\varepsilon_{i+1}} E_{\varepsilon_i-\varepsilon_{i+2}}, y_2 = E_{\varepsilon_i-\varepsilon_{i+1}} E_{\varepsilon_{i-1}-\varepsilon_{i+2}}$. We compute:

$$\begin{aligned}
y_1 x_1 \cdot v_\lambda &= E_{\varepsilon_{i-1}-\varepsilon_{i+1}} E_{\varepsilon_i-\varepsilon_{i+2}} E_{-\varepsilon_{i-1}+\varepsilon_{i+1}} E_{-\varepsilon_i+\varepsilon_{i+2}} \cdot v_\lambda \\
&= [K_{i-1,i+1}; 0][K_{i,i+2}; 0] \cdot v_\lambda \\
&\quad + E_{\varepsilon_{i-1}-\varepsilon_{i+1}} K_i^{-1} (q - q^{-1}) E_{-\varepsilon_{i-1}+\varepsilon_i} E_{\varepsilon_{i+1}-\varepsilon_{i+2}} E_{-\varepsilon_i+\varepsilon_{i+2}} \cdot v_\lambda \\
&= [K_{i-1,i+1}; 0][K_{i,i+2}; 0] \cdot v_\lambda - (q^2 - 1) K_i^{-1} E_{\varepsilon_{i-1}-\varepsilon_{i+1}} E_{-\varepsilon_{i-1}+\varepsilon_i} q K_{i+1}^{-1} E_{-\varepsilon_i+\varepsilon_{i+1}} \cdot v_\lambda \\
&= [K_{i-1,i+1}; 0][K_{i,i+2}; 0] \cdot v_\lambda - (q - q^{-1}) K_{i-1,i+2}^{-1} [K_i; 0] \cdot v_\lambda \\
&= \lambda_{i-1} \lambda_{i+1} ([c; 0]^2 - 1 + c^{-2}) \cdot v_\lambda \\
y_1 x_2 \cdot v_\lambda &= E_{\varepsilon_{i-1}-\varepsilon_{i+1}} E_{\varepsilon_i-\varepsilon_{i+2}} E_{-\varepsilon_i+\varepsilon_{i+1}} E_{-\varepsilon_{i-1}+\varepsilon_{i+2}} \cdot v_\lambda \\
&= E_{\varepsilon_{i-1}-\varepsilon_{i+1}} E_{-\varepsilon_i+\varepsilon_{i+1}} E_{\varepsilon_i-\varepsilon_{i+2}} E_{-\varepsilon_{i-1}+\varepsilon_{i+2}} \cdot v_\lambda \\
&\quad + E_{\varepsilon_{i-1}-\varepsilon_{i+1}} q^{-1} K_i^{-1} E_{\varepsilon_{i+1}-\varepsilon_{i+2}} E_{-\varepsilon_{i-1}+\varepsilon_{i+2}} \cdot v_\lambda \\
&\quad (\text{after several steps, using the fact that } \lambda_{i-1} = \pm 1) \\
&= -K_{i,i+2}^{-1} E_{\varepsilon_{i-1}-\varepsilon_{i+1}} E_{-\varepsilon_{i-1}+\varepsilon_{i+1}} \cdot v_\lambda \\
&= -K_{i,i+2}^{-1} [K_{i-1,i+1}; 0] \cdot v_\lambda \\
&= -\lambda_{i-1} \lambda_{i+1}^{-1} c^{-1} [c; 0] v_\lambda \\
y_2 x_1 \cdot v_\lambda &= E_{\varepsilon_i-\varepsilon_{i+1}} E_{\varepsilon_{i-1}-\varepsilon_{i+2}} E_{-\varepsilon_{i-1}+\varepsilon_{i+1}} E_{-\varepsilon_i+\varepsilon_{i+2}} \cdot v_\lambda \\
&= E_{\varepsilon_i-\varepsilon_{i+1}} q^{-1} K_{i-1,i+1}^{-1} E_{\varepsilon_{i+1}-\varepsilon_{i+2}} E_{-\varepsilon_i+\varepsilon_{i+2}} \cdot v_\lambda \\
&= -E_{\varepsilon_i-\varepsilon_{i+1}} K_{i-1,i+2}^{-1} E_{-\varepsilon_i+\varepsilon_{i+1}} \cdot v_\lambda \\
&= -K_{i-1,i+2}^{-1} E_{\varepsilon_i-\varepsilon_{i+1}} E_{-\varepsilon_i+\varepsilon_{i+1}} \cdot v_\lambda \\
&= -\lambda_{i-1}^{-1} \lambda_{i+1}^{-1} c^{-1} [c; 0] v_\lambda \\
y_2 x_2 \cdot v_\lambda &= E_{\varepsilon_i-\varepsilon_{i+1}} E_{\varepsilon_{i-1}-\varepsilon_{i+2}} E_{-\varepsilon_i+\varepsilon_{i+1}} E_{-\varepsilon_{i-1}+\varepsilon_{i+2}} \cdot v_\lambda \\
&= E_{\varepsilon_i-\varepsilon_{i+1}} E_{-\varepsilon_i+\varepsilon_{i+1}} E_{\varepsilon_{i-1}-\varepsilon_{i+2}} E_{-\varepsilon_{i-1}+\varepsilon_{i+2}} \cdot v_\lambda \\
&= [K_i; 0][K_{i-1,i+2}; 0] \cdot v_\lambda \\
&= \lambda_{i-1} \lambda_{i+1} [c; 0]^2 v_\lambda.
\end{aligned}$$

Therefore, by Lemma 2.1 we conclude that $[c; 0]^2([c; 0]^2 - 1) = 0$. From this we see that $c = \pm 1$, or $c = \pm q^{\pm 1}$, which finishes the proof. \square

Example: Let V be the natural representation of $U_q(\mathfrak{sl}_{n+1})$. The representations $S_q^r(V), r \in \mathbb{Z}_{\geq 0}$ and $\Lambda_q^i(V), i \in \{1, 2, \dots, n\}$ of highest weight $(q^r, 1, 1, \dots, 1)$ and $(1, 1, \dots, 1, q, 1, \dots, 1)$ with q in the i th slot, are completely pointed (see [6] for

an explicit construction). They are, up to isomorphism and tensoring with one-dimensional modules, the only finite dimensional representations in our classification (recalling that $L(q^r, 1, 1, \dots, 1) \cong L(1, 1, 1, \dots, q^r)$ from the Dynkin diagram symmetry).

Example: In this example we take $\mathbb{F} = \mathbb{Q}(q)$. Let $L(q+1)$ be the $U_q(\mathfrak{sl}_2)$ -module isomorphic to $U_q(\mathfrak{sl}_2)/J$ where J is the left $U_q(\mathfrak{sl}_2)$ ideal generated by the set $\{E, K - (q+1) \cdot 1\}$. This is a highest weight $U_q(\mathfrak{sl}_2)$ -module with highest weight vector $v_0 = 1+J$. As a $\mathbb{Q}(q)$ vector space, $L(q+1)$ has basis $\{v_k = F^{(k)} \cdot v_0 \mid k \in \mathbb{Z}_{\geq 0}\}$, where $F^{(k)} = F^k/[k]_q!$. The weight of v_k is given by the following computation:

$$K \cdot v_k = K \cdot (F^{(k)} \cdot v_0) = q^{-2k}(1+q)v_k.$$

We see that each v_k spans a one-dimensional weight space of weight $q^{-2k}(1+q)$. Also, $L(q+1)$ is irreducible, as we show by the following standard argument. Suppose that $L(q+1)$ had a proper submodule V' . Then V' would have a maximal vector of weight $q^{-2k}(q+1)$ for some $k > 0$. This maximal vector would have to be proportional to v_k for $k > 0$. But we have the following:

$$\begin{aligned} E \cdot v_k &= E \cdot (F^{(k)} \cdot v_0) \\ &= EF^{(k)} \cdot v_0 \\ &= \left(F^{(k-1)} \frac{Kq^{-k+1} - K^{-1}q^{k-1}}{q - q^{-1}} - F^{(k)}e \right) \cdot v_0 \\ &= \frac{(1+q)q^{-k+1} - (1+q)^{-1}q^{k-1}}{q - q^{-1}} F^{(k-1)} \cdot v_0 \\ &= [1+q; -k+1]v_{k-1} \end{aligned}$$

which is non-zero when $k > 0$.

We now consider the \mathbb{A} -form of $L(q+1)$, where $\mathbb{A} = \mathbb{Q}[q, q^{-1}]$. Recall that $U_{\mathbb{A}}(\mathfrak{sl}_2)$ is defined to be the \mathbb{A} -subalgebra of $U_q(\mathfrak{sl}_2)$ generated by the elements E, F, K , and $[K; 0] = (K - K^{-1})/(q - q^{-1})$. The \mathbb{A} -form of $L(q+1)$ is now the $U_{\mathbb{A}}(\mathfrak{sl}_2)$ -module $L_{\mathbb{A}}(q+1) = U_{\mathbb{A}}(\mathfrak{sl}_2) \cdot v_0$. We have:

$$v_0^{(k)} = [K; 0]^k \cdot v_0 = \left(\frac{K - K^{-1}}{q - q^{-1}} \right)^k \cdot v_0 = \left(\frac{(q+1) - (q+1)^{-1}}{q - q^{-1}} \right)^k v_0$$

are elements of $L_{\mathbb{A}}(q+1)$ for all $k > 0$ that are not in the \mathbb{A} -submodule generated by v_0 . However, these elements satisfy the following relations over \mathbb{A} :

$$(q - q^{-1})(q+1)v_0^{(k+1)} = ((q+1)^2 - 1)v_0^{(k)}.$$

Notice in this above example that the problem was not that the \mathbb{A} -form in question did not exist—indeed one can consider $U_{\mathbb{A}}(\mathfrak{g})$ acting on any $U_q(\mathfrak{g})$ -module. The problem was that passing to the $q = 1$ limit provided no information about the $U_q(\mathfrak{g})$ -module we wanted to study.

Put differently, if we define the *classical limit* of a highest weight $U_q(\mathfrak{g})$ -module $L(\lambda)$ as the $U(\mathfrak{g})$ -module $L_{\mathbb{A}}(\lambda)/(q-1)L_{\mathbb{A}}(\lambda)$, then the above equation (with $k = 0$) shows that $v_0^{(0)} \in (q-1)L_{\mathbb{A}}(q+1)$. Consequently the classical limit of the $U_q(\mathfrak{sl}_2)$ -module $L(q+1)$ is trivial. The conclusion is that the class of completely pointed $U_q(\mathfrak{g})$ -modules is richer than in the classical case, since it consists not only of q -deformations of completely pointed $U(\mathfrak{g})$ -modules.

2.2. Torsion free modules. Recall that $U_q(\mathfrak{sl}_2)$ is embedded in $U_q(\mathfrak{gl}_2)$ by adding the invertible element \bar{K}_2 satisfying the relations $\bar{K}_2 E \bar{K}_2^{-1} = q^{-1}E$, $\bar{K}_2 F \bar{K}_2^{-1} = qF$, and $K^{\pm} \bar{K}_2^{\pm} = \bar{K}_2^{\pm} K^{\pm}$ and defining $\bar{K}_1^{\pm} = (K \bar{K}_2^{-1})^{\pm}$.

Lemma 2.3. *Let V be an irreducible, torsion free, completely pointed $U_q(\mathfrak{sl}_2)$ -module and v_{λ} a weight vector in V . The action of $U_q(\mathfrak{sl}_2)$ on V can be extended to a $U_q(\mathfrak{gl}_2)$ action such that the following relations hold:*

$$FE \cdot v_{\lambda} = [\bar{K}_1; 1][\bar{K}_2; 0] \cdot v_{\lambda}.$$

Proof. Let c be the Casimir element of $U_q(\mathfrak{sl}_2)$:

$$c = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}$$

Since V is completely pointed and irreducible, c acts as a scalar $\tau \in \mathbb{F}$. Therefore, on a weight vector v_{λ} we have:

$$FE \cdot v_{\lambda} = \left(\tau - \frac{q\lambda + (q\lambda)^{-1}}{(q - q^{-1})^2} \right) v_{\lambda}.$$

Since \mathbb{F} is closed under quadratic extensions there are μ_1 and μ_2 satisfying the two equations $\tau = \frac{q\mu_1\mu_2 + (q\mu_1\mu_2)^{-1}}{(q - q^{-1})^2}$ and $\lambda = \mu_1\mu_2^{-1}$. This gives:

$$FE \cdot v_{\lambda} = \frac{q\mu_1 - (q\mu_1)^{-1}}{q - q^{-1}} \left(\frac{\mu_2 - \mu_2^{-1}}{q - q^{-1}} \right) v_{\lambda}.$$

We can make V into a $U_q(\mathfrak{gl}_2)$ -module by letting \bar{K}_2 act as μ_2 on v_{λ} and demanding that the additional relations of $U_q(\mathfrak{gl}_2)$ be satisfied, i.e. $\bar{K}_2 E \bar{K}_2^{-1} \cdot v_{\lambda} = q^{-1}E v_{\lambda}$ and $\bar{K}_2 F \bar{K}_2^{-1} \cdot v_{\lambda} = qF \cdot v_{\lambda}$. Then \bar{K}_1 acts as μ_1 which gives the desired result. \square

As before, $U_q(\mathfrak{sl}_{n+1})$ is embedded in $U_q(\mathfrak{gl}_{n+1})$ by adding the element \bar{K}_2 and defining inductively $\bar{K}_1 = K_1 \bar{K}_2$ and $\bar{K}_{i+1} = K_{i+1}^{-1} \bar{K}_i$.

Theorem 2.4. *Let V be a irreducible, torsion free, completely pointed $U_q(\mathfrak{sl}_{n+1})$ -module and v_{λ} a weight vector in V . The action of $U_q(\mathfrak{sl}_{n+1})$ on V can be extended to a $U_q(\mathfrak{gl}_{n+1})$ action such that the following relations hold:*

$$E_{-\varepsilon_i + \varepsilon_j} E_{\varepsilon_i - \varepsilon_j} \cdot v_{\lambda} = [\bar{K}_i; 1][\bar{K}_j; 0] \cdot v_{\lambda}.$$

Proof. If $n = 1$ then the result is the previous lemma. So assume $n > 1$.

Since V is completely pointed and torsion free we have, for $1 \leq i < j < k \leq n+1$, $E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda = \kappa_{ijk} E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda$, for some $\kappa_{ijk} \in \mathbb{F}$. Let $z_{ij} \in \mathbb{F}$, $1 \leq i < j \leq n+1$ be the scalars by which $E_{-\varepsilon_i + \varepsilon_j} E_{\varepsilon_i - \varepsilon_j}$ act on v_λ . Recall that $K_{ij} = \prod_{k=i}^{j-1} K_k$ and $\lambda_{ij} = \prod_{k=i}^{j-1} \lambda_k$.

We compute:

$$\begin{aligned}
0 &= E_{-\varepsilon_i + \varepsilon_j} E_{\varepsilon_i - \varepsilon_j} (E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} - \kappa_{ijk} E_{\varepsilon_i - \varepsilon_k}) \cdot v_\lambda \\
&= E_{-\varepsilon_i + \varepsilon_j} E_{\varepsilon_i - \varepsilon_j} (q^{-1} E_{\varepsilon_j - \varepsilon_k} E_{\varepsilon_i - \varepsilon_j} - (\kappa_{ijk} + 1) E_{\varepsilon_i - \varepsilon_k}) \cdot v_\lambda \\
&= E_{-\varepsilon_i + \varepsilon_j} (q^{-2} E_{\varepsilon_j - \varepsilon_k} E_{\varepsilon_i - \varepsilon_j} - (q(\kappa_{ijk} + 1) + q^{-1}) E_{\varepsilon_i - \varepsilon_k}) E_{\varepsilon_i - \varepsilon_j} \cdot v_\lambda \\
&= (-q^{-1} [K_{ij}; 0] + K_{ij}^{-1} (\kappa_{ijk} + 1)) E_{\varepsilon_j - \varepsilon_k} E_{\varepsilon_i - \varepsilon_j} \cdot v_\lambda \\
&\quad + (q^{-2} E_{\varepsilon_j - \varepsilon_k} E_{\varepsilon_i - \varepsilon_j} - (q(\kappa_{ijk} + 1) + q^{-1}) E_{\varepsilon_i - \varepsilon_k}) z_{ij} \cdot v_\lambda \\
&= (\kappa_{ijk} + 1) (-[\lambda_{ij}; 1] + (\kappa_{ijk} + 1) \lambda_{ij}^{-1}) E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda - ((q - q^{-1}) \kappa_{ijk} + q) z_{ij} E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda \\
&= \frac{\phi_{ijk} - q^{-1}}{q - q^{-1}} \left(\frac{\lambda_{ij}^{-1} \phi_{ijk} - q \lambda_{ij}}{q - q^{-1}} \right) E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda - \phi_{ijk} z_{ij} E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda \tag{*}
\end{aligned}$$

and

$$\begin{aligned}
0 &= E_{-\varepsilon_j + \varepsilon_k} E_{\varepsilon_j - \varepsilon_k} (E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} - \kappa_{ijk} E_{\varepsilon_i - \varepsilon_k}) \cdot v_\lambda \\
&= E_{-\varepsilon_j + \varepsilon_k} (q E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} + (q - \kappa_{ijk} q^{-1}) E_{\varepsilon_i - \varepsilon_k}) E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&= (-[K_{jk}; 0] - \kappa_{ijk} q^{-1} K_{jk}) E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&\quad + (q E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} + (q - \kappa_{ijk} q^{-1}) E_{\varepsilon_i - \varepsilon_k}) E_{-\varepsilon_j + \varepsilon_k} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&= \kappa_{ijk} (-[\lambda_{jk}; 1] - \kappa_{ijk} \lambda_{jk}) E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda + ((q - q^{-1}) \kappa_{ijk} + q) z_{jk} E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda \\
&= \frac{\phi_{ijk} - q}{q - q^{-1}} \left(\frac{q^{-1} \lambda_{jk}^{-1} - \phi_{ijk} \lambda_{jk}}{q - q^{-1}} \right) E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda + \phi_{ijk} z_{jk} E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda. \tag{**}
\end{aligned}$$

where $\phi_{i_1, i_2, i_3} = q + (q - q^{-1}) \kappa_{i_1, i_2, i_3}$ for arbitrary i_1, i_2, i_3 . If $\phi_{ijk} = 0$ then (*) gives the following:

$$0 = \frac{\lambda_{ij}}{(q - q^{-1})^2}$$

which is a contradiction (since every K_i acts as an invertible scalar). Therefore, $\phi_{ijk} \neq 0$. In addition, we have:

$$\begin{aligned}
z_{ij} z_{jk} v_\lambda &= E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_k - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&= E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_k - \varepsilon_j} E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&= \kappa_{ijk} E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_k - \varepsilon_j} E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda \\
&= (q^{-1} \kappa_{ijk} E_{\varepsilon_k - \varepsilon_j} E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_i - \varepsilon_k} + q^{-1} \kappa_{ijk} E_{\varepsilon_k - \varepsilon_i} E_{\varepsilon_i - \varepsilon_k}) \cdot v_\lambda \\
&= q^{-1} \kappa_{ijk} (\kappa_{ijk} z_{ik} - (q - q^{-1}) z_{ij} z_{jk}) v_\lambda + q^{-1} \kappa_{ijk} z_{ik} v_\lambda
\end{aligned}$$

whence

$$\kappa_{ijk}(\kappa_{ijk} + 1)z_{ik} = \phi_{ijk}z_{ij}z_{jk}. \quad (***)$$

From (*), (**), and (***) we deduce:

$$z_{ij} = \phi_{ijk}^{-1} \left(\frac{\phi_{ijk} - q^{-1}}{q - q^{-1}} \right) \left(\frac{\lambda_{ij}^{-1}\phi_{ijk} - q\lambda_{ij}}{q - q^{-1}} \right) \quad (2.3)$$

$$z_{jk} = -\phi_{ijk}^{-1} \left(\frac{\phi_{ijk} - q}{q - q^{-1}} \right) \left(\frac{q^{-1}\lambda_{jk}^{-1} - \phi_{ijk}\lambda_{jk}}{q - q^{-1}} \right) \quad (2.4)$$

$$z_{ik} = -\phi_{ijk}^{-1} \left(\frac{\lambda_{ij}^{-1}\phi_{ijk} - q\lambda_{ij}}{q - q^{-1}} \right) \left(\frac{q^{-1}\lambda_{jk}^{-1} - \phi_{ijk}\lambda_{jk}}{q - q^{-1}} \right). \quad (2.5)$$

This covers the case where $n = 2$, so suppose $n > 2$. If $1 \leq i < j < k < l \leq n + 1$, then (2.4) and (2.5) give:

$$\begin{aligned} & -\phi_{ijl}^{-1} \left(\frac{\phi_{ijl} - q}{q - q^{-1}} \right) \left(\frac{q^{-1}\lambda_{jl}^{-1} - \phi_{ijl}\lambda_{jl}}{q - q^{-1}} \right) \\ & = -\phi_{jkl}^{-1} \left(\frac{\lambda_{jk}^{-1}\phi_{jkl} - q\lambda_{jk}}{q - q^{-1}} \right) \left(\frac{q^{-1}\lambda_{kl}^{-1} - \phi_{jkl}\lambda_{kl}}{q - q^{-1}} \right) \end{aligned} \quad (2.6)$$

which yields:

$$(\phi_{jkl}\phi_{ijl} - \lambda_{kl}^{-2})(\lambda_{jk}^2\phi_{ijl} - \phi_{jkl}) = 0. \quad (2.7)$$

A similar argument using (2.3) and (2.5) gives:

$$(\phi_{ikl}\phi_{ijk} - \lambda_{ij}^2)(\lambda_{jk}^2\phi_{ijk} - \phi_{ikl}) = 0. \quad (2.8)$$

We compute:

$$\begin{aligned} \kappa_{jkl}E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda &= E_{\varepsilon_j - \varepsilon_k}E_{\varepsilon_k - \varepsilon_l} \cdot v_\lambda \\ &= (-E_{\varepsilon_j - \varepsilon_l} + q^{-1}E_{\varepsilon_k - \varepsilon_l}E_{\varepsilon_j - \varepsilon_k}) \cdot v_\lambda \end{aligned}$$

hence $q(\kappa_{jkl} + 1)E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda = E_{\varepsilon_k - \varepsilon_l}E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda$. Therefore we have:

$$\begin{aligned} q\kappa_{ijl}(\kappa_{jkl} + 1)E_{\varepsilon_i - \varepsilon_l} \cdot v_\lambda &= q(\kappa_{jkl} + 1)E_{\varepsilon_i - \varepsilon_j}E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda \\ &= E_{\varepsilon_k - \varepsilon_l}E_{\varepsilon_i - \varepsilon_j}E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\ &= \kappa_{ijk}E_{\varepsilon_k - \varepsilon_l}E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda \\ &= q\kappa_{ijk}(\kappa_{ikl} + 1)E_{\varepsilon_i - \varepsilon_l} \cdot v_\lambda. \end{aligned}$$

Therefore, since $E_{\varepsilon_i - \varepsilon_l} \cdot v_\lambda \neq 0$, we have:

$$\kappa_{ijl}(\kappa_{jkl} + 1) = \kappa_{ijk}(\kappa_{ikl} + 1) \quad (2.9)$$

and analogously:

$$\kappa_{jil}(\kappa_{ikl} + 1) = \kappa_{jik}(\kappa_{jkl} + 1). \quad (2.10)$$

Now we derive more relations for κ_{i_1, i_2, i_3} :

$$\begin{aligned}
E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda &= (-E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} + q^{-1} E_{\varepsilon_j - \varepsilon_l} E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k}) \cdot v_\lambda \\
&= (-E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} + q^{-1} \kappa_{ijk} E_{\varepsilon_j - \varepsilon_l} E_{\varepsilon_i - \varepsilon_k}) \cdot v_\lambda \\
&= (-q^{-1}(q - q^{-1}) \kappa_{ijk} - 1) E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&\quad + q^{-1} \kappa_{ijk} E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda
\end{aligned}$$

and

$$\begin{aligned}
E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda &= q^{-1} E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_k} E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda \\
&= (-q^{-1} E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_j - \varepsilon_l} + q^{-2} E_{\varepsilon_j - \varepsilon_k} E_{\varepsilon_i - \varepsilon_j} E_{\varepsilon_j - \varepsilon_l}) \cdot v_\lambda \\
&= (-q^{-1} E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_j - \varepsilon_l} + q^{-2} \kappa_{ijl} E_{\varepsilon_j - \varepsilon_k} E_{\varepsilon_i - \varepsilon_l}) \cdot v_\lambda.
\end{aligned}$$

Therefore:

$$\begin{aligned}
0 &= \kappa_{jkl}(q^{-1}(q - q^{-1}) \kappa_{ijk} + 1 + q^{-2} \kappa_{ijl}) E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&\quad - q^{-1} \kappa_{jkl}(\kappa_{ijk} + 1) E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda \\
&= \kappa_{jkl}(q^{-1}(q - q^{-1}) \kappa_{ijk} + 1 + q^{-2} \kappa_{ijl}) E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&\quad - q^{-1}(\kappa_{ijk} + 1) E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_j - \varepsilon_l} E_{\varepsilon_k - \varepsilon_l} \cdot v_\lambda \\
&= \kappa_{jkl}(q^{-1}(q - q^{-1}) \kappa_{ijk} + 1 + q^{-2} \kappa_{ijl}) E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda \\
&\quad - \kappa_{ikl}(\kappa_{ijk} + 1) E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda
\end{aligned}$$

which gives:

$$\kappa_{jkl}(q^{-1}(q - q^{-1}) \kappa_{ijk} + 1 + q^{-2} \kappa_{ijl}) = \kappa_{ikl}(\kappa_{ijk} + 1). \quad (2.11)$$

Subtracting (2.9) from the above we see:

$$\kappa_{jkl}(q^{-1}(q - q^{-1}) \kappa_{ijk} + 1 - q^{-1}(q - q^{-1}) \kappa_{ijl}) - \kappa_{ijl} = \kappa_{ikl} - \kappa_{ijk} \quad (2.12)$$

$$(q^{-1} \phi_{jkl} - 1)(\phi_{ijk} - \phi_{ijl}) + \phi_{jkl} - \phi_{ijl} = \phi_{ikl} - \phi_{ijk} \quad (2.13)$$

$$q^{-1} \phi_{jkl}(\phi_{ijk} - \phi_{ijl}) + \phi_{jkl} - \phi_{ikl} = 0. \quad (2.14)$$

We repeat the same argument with the indices i and j transposed to obtain:

$$\begin{aligned}
E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda &= (-q^{-1} K_{ij}^{-1} E_{\varepsilon_j - \varepsilon_l} E_{\varepsilon_i - \varepsilon_k} + E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_i - \varepsilon_k}) \cdot v_\lambda \\
&= (-q^{-1} \lambda_{ij}^{-1} E_{\varepsilon_j - \varepsilon_l} E_{\varepsilon_i - \varepsilon_k} + \kappa_{jik} E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k}) \cdot v_\lambda \\
&= -q^{-1} \lambda_{ij}^{-1} E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda \\
&\quad + (\kappa_{jik} + q^{-1}(q - q^{-1}) \lambda_{ij}^{-1}) E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda
\end{aligned}$$

and

$$\begin{aligned}
E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_i - \varepsilon_k} \cdot v_\lambda &= q^{-1} E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_i - \varepsilon_l} \cdot v_\lambda \\
&= (-q^{-2} K_{ij}^{-1} E_{\varepsilon_j - \varepsilon_k} E_{\varepsilon_i - \varepsilon_l} + q^{-1} E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_j - \varepsilon_i} E_{\varepsilon_i - \varepsilon_l}) \cdot v_\lambda \\
&= (-q^{-2} \lambda_{ij}^{-1} E_{\varepsilon_i - \varepsilon_l} E_{\varepsilon_j - \varepsilon_k} + q^{-1} \kappa_{jil} E_{\varepsilon_i - \varepsilon_k} E_{\varepsilon_j - \varepsilon_l}) \cdot v_\lambda.
\end{aligned}$$

Hence:

$$\begin{aligned}
0 &= \kappa_{ikl}(\kappa_{jik} + \lambda_{ij}^{-1})E_{\varepsilon_i - \varepsilon_l}E_{\varepsilon_j - \varepsilon_k} \cdot v_\lambda - q^{-1}\kappa_{ikl}(\kappa_{jil} + \lambda_{ij}^{-1})E_{\varepsilon_i - \varepsilon_k}E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda \\
&= (\kappa_{jik} + \lambda_{ij}^{-1})E_{\varepsilon_j - \varepsilon_k}E_{\varepsilon_i - \varepsilon_k}E_{\varepsilon_k - \varepsilon_l} \cdot v_\lambda - q^{-1}\kappa_{ikl}(\kappa_{jil} + \lambda_{ij}^{-1})E_{\varepsilon_i - \varepsilon_k}E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda \\
&= q^{-1}\kappa_{jkl}(\kappa_{jik} + \lambda_{ij}^{-1})E_{\varepsilon_i - \varepsilon_k}E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda - q^{-1}\kappa_{ikl}(\kappa_{jil} + \lambda_{ij}^{-1})E_{\varepsilon_i - \varepsilon_k}E_{\varepsilon_j - \varepsilon_l} \cdot v_\lambda
\end{aligned}$$

which gives:

$$\kappa_{jkl}(\kappa_{jik} + \lambda_{ij}^{-1}) = \kappa_{ikl}(\kappa_{jil} + \lambda_{ij}^{-1}). \quad (2.15)$$

Subtracting (2.10) from the above gives:

$$\lambda_{ij}^{-1}\kappa_{jkl} - \kappa_{jik} = \lambda_{ij}^{-1}\kappa_{ikl} - \kappa_{jil} \quad (2.16)$$

$$\lambda_{ij}^{-1}(\kappa_{jkl} - \kappa_{ikl}) = \kappa_{jik} - \kappa_{jil}. \quad (2.17)$$

We compute:

$$\begin{aligned}
0 &= (E_{\varepsilon_k - \varepsilon_j}E_{\varepsilon_j - \varepsilon_i}E_{\varepsilon_i - \varepsilon_k} - \kappa_{jik}E_{\varepsilon_k - \varepsilon_j}E_{\varepsilon_j - \varepsilon_i}) \cdot v_\lambda \\
&= (\kappa_{ijk}z_{ik} - (q - q^{-1})z_{ij}z_{jk} - \kappa_{jik}z_{jk})v_\lambda \\
&= ((\kappa_{ijk} + 1)^{-1}\phi_{ijk} - (q - q^{-1}))z_{ij}z_{jk}v_\lambda - \kappa_{jik}z_{jk}v_\lambda.
\end{aligned}$$

Hence, using (2.3)–(2.5) we see:

$$\kappa_{jik} = \frac{q^{-1}\lambda_{ij}^{-1} - \lambda_{ij}\phi_{ijk}^{-1}}{q - q^{-1}} \quad (2.18)$$

and similarly,

$$\kappa_{jil} = \frac{q^{-1}\lambda_{ij}^{-1} - \lambda_{ij}\phi_{ijl}^{-1}}{q - q^{-1}}. \quad (2.19)$$

Using the above in (2.17) gives:

$$\lambda_{ij}^{-1}\phi_{ijk}\phi_{ijl}(\phi_{jkl} - \phi_{ikl}) = \lambda_{ij}(\phi_{ijk} - \phi_{ijl}). \quad (2.20)$$

Equations (2.7), (2.8), (2.14), and (2.20) have a unique simultaneous solution for ϕ_{ijl} , ϕ_{ikl} , ϕ_{jkl} in terms of ϕ_{ijk} , namely:

$$\phi_{ijl} = \phi_{ijk}, \phi_{ikl} = \phi_{jkl} = \lambda_{jk}^2\phi_{ijk}. \quad (2.21)$$

From these relations and equations (2.3)–(2.5) we see that all the z_{ij} are determined by ϕ_{123} . Using that \mathbb{F} is closed under quadratic extensions, we choose $\mu_2 = \pm(q\phi_{123})^{-1/2}$ and obtain the desired result: $z_{ij} = [\mu_i; 1][\mu_j; 0]$. \square

We remark that analogues of equations (2.14) and (2.20) were found in [2] and in the $q = 1$ limit they imply the first two identities in (2.21), but in our case all four equations are needed.

2.3. Modules with torsion. We now set out to prove an extension of the Theorem 2.4 to the torsion case. Before we do so, let us introduce some notation. Let V be an irreducible $U_q(\mathfrak{sl}_{n+1})$ -module. Let $N = \{\beta \in \Phi \mid \forall v \in V, \exists k > 0 \text{ such that } E_\beta^k \cdot v = 0\}$, $T = \{\beta \in \Phi \mid \forall v \in V, E_\beta \cdot v \neq 0\}$, $N_s = N \cap (-N)$, $T_s = T \cap (-T)$, $N_a = N \setminus N_s$ and $T_a = T \setminus T_s$. Finally, define $V^+ = \{v \in V \mid \forall \beta \in N_a \cup N_s^+, E_\beta \cdot v = 0\}$ where $N_s^+ = \Phi^+ \cap N_s$. Using that q is not a root of unity it is easy to show that $\Phi = N \cup T$. The following is an analogue of (4.6) and (4.12) in [4], but in our proof for completely-pointed modules we avoid working with the center of $U_q(\mathfrak{g})$.

Proposition 2.5. *Let V be an irreducible completely-pointed $U_q(\mathfrak{sl}_{n+1})$ weight module. Then N and T are closed subsets of Φ .*

Proof. Let $\alpha, \beta \in N$ be such that $\alpha + \beta \in \Phi$. Then there exist $0 \neq v^+ \in V$ such that $E_\alpha \cdot v^+ = 0$ and $s \in \mathbb{Z}_{>0}$ such that $E_\beta^s \cdot v^+ = 0$ and $E_\beta^{s-1} \cdot v^+ \neq 0$. Note that equations (1.1)–(1.4) imply $KE_{\alpha+\beta} = \pm(q^j E_\alpha E_\beta - q^k E_\beta E_\alpha)$ for some $j, k \in \{-1, 0, 1\}$, and invertible $K \in \mathbb{F}[K_1^{\pm 1}, K_2^{\pm 1}, \dots, K_n^{\pm 1}]$. Using this, we compute:

$$\begin{aligned} 0 &= E_\alpha E_\beta^s \cdot v^+ = \pm \sum_{i=0}^{s-1} q^{k_i} E_\beta^i K E_{\alpha+\beta} E_\beta^{s-i-1} \cdot v^+ \\ &= \pm \left(\sum_{i=0}^{s-1} q^{k_i + r_i} \right) K E_{\alpha+\beta} \cdot (E_\beta^{s-1} \cdot v^+) \end{aligned} \quad (2.22)$$

where $k_i, r_i \in \mathbb{Z}, i \in \{1, 2, \dots, n\}$ are increasing or decreasing sequences. Therefore $\alpha + \beta \in N$, which gives that N is closed.

Now let $\alpha, \beta \in T$ be such that $\alpha + \beta \in \Phi$. Since $\alpha, \beta \in T$ we have $E_\alpha^k \cdot v_\lambda \neq 0$ and $E_\beta^r \cdot v_\lambda \neq 0$ for all $k, r \in \mathbb{Z}_{\geq 0}$ and all weights $\lambda \in (\mathbb{F}^\times)^{n+1}$, hence $q^{\mathbb{Z}_{\geq 0}(\alpha+\beta)} \text{supp}(V) \subseteq \text{supp}(V)$. If $E_{\alpha+\beta} \in N$ then, given $\lambda \in \text{wt}(V)$, we have an infinite sequence of vectors $v_i \in V_{q^{m_i}(\alpha+\beta)\lambda}$ where $m_i \in \mathbb{Z}_{\geq 0}$ is an increasing sequence such that $E_{\alpha+\beta} \cdot v_i = 0$. We compute:

$$\begin{aligned} 0 &= E_{\alpha+\beta} \cdot v_i \\ &= c E_{\alpha+\beta} (E_{-\alpha-\beta})^{m_{i+1}-m_i} \cdot v_{i+1} \\ &\quad (\text{for some } c \in \mathbb{F} \setminus \{0\} \text{ since } V \text{ is completely pointed}) \\ &= c \left(\sum_{i=0}^{m_{i+1}-m_i-1} (E_{-\alpha-\beta})^i [K_{\alpha+\beta}; 0] (E_{-\alpha-\beta})^{m_{i+1}-m_i-i} \right) \cdot v_{i+1} \\ &= c [m_{i+1} - m_i]_q [K_{\alpha+\beta}; m_{i+1} - m_i - 1] (E_{-\alpha-\beta})^{m_{i+1}-m_i-1} \cdot v_{i+1}. \end{aligned}$$

Since $c(E_{-\alpha-\beta})^{m_{i+1}-m_i-1} \cdot v_{i+1} \neq 0$ and $m_{i+1} > m_i$ we see that $[K_{\alpha+\beta}; m_{i+1} - m_i - 1] \cdot v_{i+1} = 0$. Therefore, letting $\lambda_{\alpha+\beta} \in \mathbb{F} \setminus \{0\}$ be the eigenvalue of $K_{\alpha+\beta}$ on v_{i+1} , we see that $\lambda_{\alpha+\beta} = \pm q^{-(m_{i+1}-m_i-1)}$ —a non-positive integral power of q times ± 1 . However, since $K_{\alpha+\beta} \cdot v_{i+j} = q^{2(m_j-m_{i+1})} \lambda_{\alpha+\beta} v_{i+j}$ and m_j is an increasing sequence of integers, for some $j^* > i$ we must have $K_{\alpha+\beta}$ acting as a non-negative power of

q times ± 1 on v_{j^*} . This j^* is therefore the highest index in the sequence of m_j , contrary to it being an infinite sequence. Therefore $\alpha + \beta \notin N$ and must be in T since $\Phi = N \cup T$. \square

As a corollary, we have that N_s and T_s are root subsystems of Φ .

Lemma 2.6. *If \mathfrak{g} is simply-laced, then there exists a base B of $\Phi(\mathfrak{g})$ such that $N_a \subseteq \Phi_B^+$, and every $\alpha \in B \setminus N_a$ is a positive root (with respect to the usual base of Φ).*

Proof. Lemma 4.7 (i) of [1] proves the existence of a base B of $\Phi = \Phi(\mathfrak{g})$ such that $N_a^+ \subseteq \Phi_B^+$. We may apply their result since the proof only uses results on root subsystems that satisfy the same hypotheses as in our case. We show how to choose a new base B_n satisfying the same condition, but with every $\alpha \in B \setminus N_a$ positive with respect to the usual base, B_u . By the previous proposition we see that N_s and T_s are root subsystems of Φ . Let W_N and W_T be the Weyl group of the root subsystems N_s and T_s respectively. We may choose a base B_p of $N_s \cup T_s$ such that $(N_s \cup T_s) \cap B_p$ is contained in $\Phi_{B_u}^+$. It is a well known fact of finite root systems (see [8, Section 10.1]) that the Weyl group permutes bases. Let $w \in W_T \times W_N$ be the Weyl group element taking $B \cap (N_s \cup T_s)$ to B_p . We want to show that w preserves N_a . Let $\alpha \in N_a$ and $\beta \in B \cap (N_s \cup T_s)$ be given. We have $r_\beta(\alpha) = \alpha - \langle \beta, \alpha \rangle \beta \in \Phi$. Since \mathfrak{g} is simply laced, $\langle \beta, \alpha \rangle = 0$ or -1 . In the first case, $r_\beta(\alpha) = \alpha \in N_a$. Otherwise, $r_\beta(\alpha) = \alpha + \beta$. By (ii) of the above lemma, we have $\alpha + \beta \in N_a$. Hence, $w(\alpha) \in N_a$, and $w(B)$ is the desired basis. \square

We let $U_q(\mathfrak{g}_{N_s})$ (resp. $U_q(\mathfrak{n}_{N_s})$) denote the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\alpha | \alpha \in N_s\}$ and (resp. $\{E_\alpha | \alpha \in N_s^+\}$).

Proposition 2.7. *Suppose V is an irreducible completely-pointed $U_q(\mathfrak{g})$ -module. Then the following hold:*

- (1) *if $N \neq \emptyset$ then the set V^+ is non-zero, i.e. there exists $v^+ \in V \setminus \{0\}$ such that $E_\alpha \cdot v^+ = 0$ for $\alpha \in N_s^+ \cup N_a$,*
- (2) *either $B \cap T$ is empty or it corresponds to a connected part of the Dynkin diagram of Φ_B where B is the basis of the previous lemma.*

Proof. Proof of (1). Let $v^* \in V \setminus \{0\}$ and $\{\beta_1, \beta_2, \dots, \beta_l\} = N \cap B$ where B is the base of Φ given in the previous lemma. Then there exist $r_j \in \mathbb{Z}_{>0}$, $j \in \{1, 2, \dots, l\}$ such that $E_{\beta_j}^{r_j} \prod_{k=j+1}^l E_{\beta_k}^{r_k-1} \cdot v^* = 0$ and r_j is minimal that this occurs. Let $v^+ = \prod_{k=1}^l E_{\beta_k}^{r_k-1} \cdot v^*$. Then $v^+ \neq 0$ and $E_{\beta_j} \cdot v^+ = 0$, $j \in \{1, 2, \dots, l\}$. Since V is completely pointed, we have $E_\alpha E_\beta \cdot v_\lambda = c E_\beta E_\alpha \cdot v_\lambda$ for some $c \in \mathbb{F}$ whenever v_λ is a weight vector such that $E_\beta \cdot v_\lambda \neq 0$, hence $E_{\beta_k} \cdot v^+ = 0$ for all $k \in \{1, 2, \dots, l\}$. Now, let $\gamma \in N_a \cup N_s^+$. Then $\gamma \in \Phi_B^+$ hence it can be written as a positive integral combination of elements of B . At least one of these simple roots must be in $N \cap B$ since otherwise $\gamma \in T$ which is a contradiction. Let $\alpha, \beta \in B$ such that $E_\beta \cdot v^+ = 0$

and $\alpha + \beta \in \Phi$. We compute

$$\begin{aligned} E_{\alpha+\beta} \cdot v^+ &= \pm K(q^j E_\beta E_\alpha - q^k E_\alpha E_\beta) \cdot v^+ \\ &= \pm K(q^j - cq^k) E_\beta E_\alpha \cdot v^+ \\ &= 0 \end{aligned}$$

for some $j, k \in \{-1, 0, 1\}$, $K \in \mathbb{F}[K_1^{\pm 1}, K_2^{\pm 1}, \dots, K_n^{\pm 1}]$ and $c \in \mathbb{F}$. Let $\beta_i \in N \cap B$ be one of the simple roots in the decomposition of γ . Then there exists a sequence of roots $\gamma_0 = \beta_i, \gamma_1, \dots, \gamma_r = \gamma$ with $\gamma_{i+1} - \gamma_i \in B$. Since $E_{\beta_i} \cdot v^+ = 0$ we may use induction on the sequence of γ_k to see that $E_\gamma \cdot v^+ = 0$. Therefore v^+ is the vector we are looking for.

The proof of (2) is similar to Lemma 4.9 of [1] and we leave it to the reader. \square

We are now ready to prove:

Theorem I. *Let V be an irreducible, infinite-dimensional, completely-pointed $U_q(\mathfrak{sl}_{n+1})$ -module and let $v^+ \in V^+$ be given. Then the action of $U_q(\mathfrak{sl}_{n+1})$ on V can be extended to a $U_q(\mathfrak{gl}_{n+1})$ action such that the following relations hold:*

$$E_{-\varepsilon_i + \varepsilon_j} E_{\varepsilon_i - \varepsilon_j} \cdot v^+ = [\bar{K}_i; 1][\bar{K}_j; 0] \cdot v^+.$$

Also, we have

$$F_i E_i \cdot v_\lambda = [\bar{K}_i; 1][\bar{K}_{i+1}; 0] \cdot v_\lambda$$

for any weight vector $v_\lambda \in V$.

Proof. There are three cases: the extreme cases $T_s = \emptyset, T_s = \Phi$ and the intermediate case $\emptyset \subsetneq T_s \subsetneq \Phi$.

Case i: $T_s = \emptyset$. In this case, V is a highest weight module, hence, by Proposition 2.2, is isomorphic to $L(\lambda)$ where $\lambda = (c, \pm 1, \pm 1, \dots, \pm 1), (\pm 1, \pm 1, \dots, \pm 1, c)$, with c not a positive integer power of q , or $(\pm 1, \pm 1, \dots, c, c^{-1}q^{-1}, \dots, \pm 1)$. As before, let $z_{ij} \in \mathbb{F}, 1 \leq i < j \leq n+1$ be the scalars by which $E_{-\varepsilon_i + \varepsilon_j} E_{\varepsilon_i - \varepsilon_j}$ act on v_λ . We have $z_{ij} = 0$ for $i < j \in \{1, 2, \dots, n+1\}$. In the first case, we may choose $\mu_1 = c$, and $\mu_i = \pm 1, 1 < i \leq n+1$. In the second case $\mu_{n+1} = q^{-1}c^{-1}$ and $\mu_i = \pm q^{-1}, 1 \leq i < n+1$ gives the desired result. In the third case, if $\lambda_k = c$ then we choose $\mu_i = \pm q^{-1}, 1 \leq i < k, \mu_k = q^{-1}c^{-1}$, and $\mu_i = \pm 1, k < i \leq n+1$ satisfies the conditions of the theorem.

Case ii: $T_s = \Phi$. In this case, V is a torsion free module. By Theorem 2.4 the conclusion holds for some weight vector v_λ . As in the last part of the next case one shows that then it holds for any weight vector.

Case iii: $\emptyset \subsetneq T_s \subsetneq \Phi$. Let $\beta_1, \beta_2, \dots, \beta_n$ be a base of Φ such that $N_a \subseteq \Phi_B^+$ with $\beta_i \in T_s$ for $i \in \{k, k+1, \dots, l\}$. Let v^+ be an invariant vector of weight λ for $E_\alpha, \alpha \in N_a \cup N_s^+$. Using Theorem 2.4 we can choose $\mu_{s_i}, i \in \{k, k+1, \dots, l+1\}$ such that $z_{s_i, s_j} = [\mu_{s_i}; 1][\mu_{s_j}; 0]$ and $\lambda_{s_i, s_j} = \mu_{s_i} \mu_{s_j}^{-1}$ for $i, j \in \{k, k+1, \dots, l+1\}$. For $\beta \in N_s^+ \cup N_a$ we have $E_{-\beta} E_\beta \cdot v_\lambda = 0$. We need to show that this choice

of μ_{s_i} induces a choice $\mu'_i, i \in \{1, 2, \dots, n+1\}$ satisfying $z_{ij} = [\mu'_i; 1][\mu'_j; 0]$ and $\lambda_{ij} = \mu'_i \mu_j^{-1}$ for $i \neq j$.

Let $\beta = \varepsilon_t - \varepsilon_u \in N_a, \beta' = \varepsilon_u - \varepsilon_v \in T_s^+$. We have $z_{ut}v^+ = E_\beta E_{-\beta}v^+ = [\lambda_{tu}; 0]v^+$. Since $-\beta, -\beta' \in T$ we can define κ_{vut} similarly to Theorem 2.4 by $E_{-\beta}E_{-\beta'} \cdot v^+ = \kappa_{vut}E_{-\beta-\beta'} \cdot v^+$. By similar computations to those used to prove (2.3)–(2.5) we obtain the following for $1 \leq i < j < k \leq n+1$:

$$(q^{-1} - (q - q^{-1})\kappa_{kji})z_{kj} = (\kappa_{kji} + 1)(\lambda_{jk}^{-1}\kappa_{kji} + q^{-1}[\lambda_{jk}; 0]) \quad (2.23)$$

$$(q^{-1} - (q - q^{-1})\kappa_{kji})z_{ji} = \kappa_{kji}(\lambda_{ij}\kappa_{kji} - [\lambda_{ij}; -1]) \quad (2.24)$$

$$(q^2\lambda_{ij} + (q - q^{-1})\kappa_{kij})z_{ki} = (\kappa_{kij} + q\lambda_{ij})(q^{-1}\lambda_{jk}\kappa_{kij} + q[\lambda_{ik}; 0]) \quad (2.25)$$

$$(q^2\lambda_{ij} + (q - q^{-1})\kappa_{kij})z_{ij} = \kappa_{kij}(q^{-1}\kappa_{kij} + [\lambda_{ij}; 1]) \quad (2.26)$$

$$(q^{-2}\lambda_{jk} - (q - q^{-1})\kappa_{ikj})z_{ik} = (\kappa_{ikj} + q^{-1}\lambda_{jk})(q\lambda_{ij}\kappa_{ikj} - q^{-1}[\lambda_{ik}; 0]) \quad (2.27)$$

$$(q^{-2}\lambda_{jk} - (q - q^{-1})\kappa_{ikj})z_{kj} = \kappa_{ikj}(q\kappa_{ikj} - [\lambda_{jk}; -1]) \quad (2.28)$$

$$(q\lambda_{jk}^{-1} + (q - q^{-1})\kappa_{jki})z_{jk} = (\kappa_{jki} + \lambda_{jk}^{-1})(\kappa_{jki} - q[\lambda_{jk}; 0]) \quad (2.29)$$

$$(q\lambda_{jk}^{-1} + (q - q^{-1})\kappa_{jki})z_{ki} = \kappa_{jki}(\lambda_{ij}^{-1}\kappa_{jki} - [\lambda_{ik}; -1]) \quad (2.30)$$

$$(q^{-1}\lambda_{ij}^{-1} - (q - q^{-1})\kappa_{jik})z_{ji} = (\kappa_{jik} + \lambda_{ij}^{-1})(\kappa_{jik} + q^{-1}[\lambda_{ij}; 0]) \quad (2.31)$$

$$(q^{-1}\lambda_{ij}^{-1} - (q - q^{-1})\kappa_{jik})z_{ik} = \kappa_{jik}(\lambda_{jk}^{-1}\kappa_{jik} + [\lambda_{ik}; 1]) \quad (2.32)$$

The proof now depends on the order of t, u, v . We consider the case $t < u < v$, with similar arguments giving the other cases. Putting $i = t, j = u, k = v$, equations (2.23) and (2.24) hold. Since $z_{ji} = [\lambda_{ij}; 0]$ this gives:

$$(q^{-1} - (q - q^{-1})\kappa_{kji})[\lambda_{ij}; 0] = \kappa_{kji}(\lambda_{ij}\kappa_{kji} - [\lambda_{ij}; -1])$$

which may be solved to give $\kappa_{kji} = -1$ or $q^{-1}\lambda_{ij}^{-1}[\lambda_{ij}; 0]$. But, if $\kappa_{kji} = -1$ then (2.23) gives $z_{kj} = 0$ contradicting $\varepsilon_j - \varepsilon_k \in T_s$. Hence $\kappa_{kji} = q^{-1}\lambda_{ij}^{-1}[\lambda_{ij}; 0]$. Inserting this in (2.23) gives:

$$\begin{aligned} q^{-1}\lambda_{ij}^{-2}z_{kj} &= \frac{-q^{-1}\lambda_{ij}^{-2} + q}{q - q^{-1}} \left(\frac{-q^{-1}\lambda_{jk}^{-1}\lambda_{ij}^{-2} + q^{-1}\lambda_{jk}}{q - q^{-1}} \right) \\ z_{kj} &= \frac{q\lambda_{ij} - q^{-1}\lambda_{ij}^{-1}}{q - q^{-1}} \left(\frac{\lambda_{ij}\lambda_{jk} - (\lambda_{ij}\lambda_{jk})^{-1}}{q - q^{-1}} \right). \end{aligned}$$

In light of Theorem 2.4, for the root subsystem T_s we have $z_{kj} = [\mu_k; 1][\mu_j; 0], \lambda_{jk} = \mu_j\mu_k^{-1}$ for some $\mu_k, \mu_j \neq 0$ in \mathbb{F} (using that \mathbb{F} is closed under quadratic extensions). Therefore $\lambda_{ij} = \pm\mu_k$ or $\pm q^{-1}\mu_j^{-1}$. Similarly, for any order of t, u, v we have $\lambda_{tu} = \pm\mu_v$ or $\pm q^{-1}\mu_u^{-1}$. Defining $\beta = \varepsilon_t - \varepsilon_u \in T_s, \beta' = \varepsilon_u - \varepsilon_v \in N_a$ we similarly see $\lambda_{uv} = \pm\mu_u$ or $\pm q^{-1}\mu_t^{-1}$ for some $\mu_t, \mu_u \neq 0$ such that $z_{tu} = [\mu_t; 1][\mu_u; 0], \lambda_{tu} = \mu_t\mu_u^{-1}$.

For $\varepsilon_t - \varepsilon_u$ and $\varepsilon_u - \varepsilon_v \in N_s^+ \cup N_a$, by computations similar to (2.1), $\lambda_{tu}\lambda_{uv} = \pm q^{-1}, \pm\lambda_{uv}$ or $\pm\lambda_{tu}$. Since $\varepsilon_{s_k} - \varepsilon_{s_{k+1}} \in T_s^+$ and $\varepsilon_{s_{k-2}} - \varepsilon_{s_k} \in N_a$, we have

$\lambda_{s_{k-2}, s_{k-1}} \lambda_{s_{k-1}, s_k} = \pm q^{-1} \mu_{s_k}^{-1}$ or $\pm \mu_{s_{k+1}}$. If $\lambda_{s_{k-2}, s_{k-1}} \lambda_{s_{k-1}, s_k} = \pm q^{-1}$, then $\mu_{s_k} = \pm 1$ or $\mu_{s_{k+1}} \pm q^{-1}$, each of which imply $z_{s_{k+1}, s_k} = 0$ contradicting $\varepsilon_{s_k} - \varepsilon_{s_{k+1}} \in T_s$. Therefore $\lambda_{s_{k-2}, s_{k-1}} = \pm 1$. In the following, we let $I_1 = \{1, 2, \dots, k-1\}$, $I_2 = \{k, k+1, \dots, l+1\}$, $I_3 = \{l+2, l+3, \dots, n+1\}$. By (2.2), we have $\lambda_{s_i, s_j} = \pm 1$ for $i < j \in I_1$. Similarly, $\lambda_{s_i, s_j} = \pm 1$ for $i < j \in I_3$.

Now, let $i \in I_2, j \in I_3$. By construction $\varepsilon_{s_i} - \varepsilon_{s_{l+1}} \in T_s^+$ and $\varepsilon_{s_{l+1}} - \varepsilon_{s_j} \in N_a$, and we have $\lambda_{s_{l+1}, s_j} = \pm \mu_{s_{l+1}}$ or $\pm q^{-1} \mu_{s_i}^{-1}$. If the first case holds for some $i \in I_2$ then we keep $\mu'_{s_i} = \mu_{s_i}, i \in I_2$. Otherwise, it is the case that $\lambda_{s_{l+1}, s_j} = \pm q^{-1} \mu_{s_r}^{-1}$ for all $r \in I$, which implies that $\mu_{s_r} = c$ for $r \in I$. Now, having $\varepsilon_{s_i} - \varepsilon_{s_l} \in T_s^+$ and $\varepsilon_{s_l} - \varepsilon_{s_j} \in N_a$ gives $\lambda_{s_l, s_{l+1}} \lambda_{s_{l+1}, s_j} = \pm c$ or $\pm q^{-1} c^{-1}$, hence $\mu_{s_{l+1}} = \pm q^{-1} c^{-1}$ or $\pm c$. In the first case, we we keep $\mu'_{s_i} = \mu_{s_i}, i \in I_2$. In the second case, we make a change of variables $\mu'_{s_i} = q^{-1} \mu_{s_i}^{-1}, k \leq i \leq l+1$ giving $[\mu_{s_i}; 1][\mu_{s_{i+1}}; 0] = [\mu'_{s_{i+1}}; 1][\mu'_{s_i}; 0]$ and $\mu_{s_i} \mu_{s_{i+1}}^{-1} = \mu'_{s_{i+1}} \mu_{s_i}^{-1}$, hence these relations are preserved. In each case above, with our choice of μ'_{s_i} , we have $\lambda_{s_{l+1}, s_j} = \pm \mu'_{s_{l+1}}$. Therefore $\lambda_{s_i, s_j} = \lambda_{s_i, s_{l+1}} \lambda_{s_{l+1}, s_j} = \pm \mu'_i$ for $i \in I_2$ and $j \in I_3$. Finally, for $i \in I_1$ and $j \in I_2$, since $\lambda_{s_i, s_j} \neq \pm 1$ and $\lambda_{s_j, s_{l+2}} \neq \pm 1$, we must have $\lambda_{s_i, s_j} \lambda_{s_j, s_{l+2}} = \pm q^{-1}$. Therefore $\lambda_{s_i, s_j} = \pm q^{-1} \mu'_j$.

Summarizing, we have:

$$\lambda_{s_i, s_j} = \begin{cases} \pm q^{-1} \mu'_j{}^{-1} & \text{for } i \in I_1, j \in I_2 \\ \pm \mu'_i & \text{for } i \in I_2, j \in I_3 \\ \pm \mu'_i \mu'_j{}^{-1} & \text{for } i < j \in I_2, \\ \pm q^{-1} & \text{for } i \in I_1, j \in I_3 \\ \pm 1 & \text{otherwise.} \end{cases}$$

Therefore, we can put $\mu'_i = \pm q^{-1}$ for $i \in I_1$ and $\mu'_j = \pm 1$ for $j \in I_3$. Hence, for $i < j \in \{1, 2, \dots, n+1\}$ we have $\lambda_{s_i, s_j} = \mu'_{s_i} \mu_{s_j}^{-1}$ and $z_{s_i, s_j} = [\mu'_{s_i}; 1][\mu'_{s_j}; 0]$.

It follows that for $1 \leq i \leq n+1$ we have $F_i E_i \cdot v_\lambda = [\mu_i; 1][\mu_{i+1}; 0] \cdot v_\lambda = [\bar{K}_i; 1][\bar{K}_{i+1}; 0] \cdot v_\lambda$ for some weight vector $v_\lambda \in V$. It remains to show that the same holds for any weight vector in V . Let $\varepsilon_j - \varepsilon_k \in T \cup N_s^-$ be given, and suppose $j < k$ (the case $k < j$ is symmetric). If $i \neq j-1, j, k-1$ or k then $E_{\varepsilon_j - \varepsilon_k}$ commutes with $F_i E_i, \bar{K}_i$ and \bar{K}_{i+1} , hence we have

$$F_i E_i E_{\varepsilon_j - \varepsilon_k} \cdot v^+ = [\bar{K}_i; 1][\bar{K}_{i+1}; 0] E_{\varepsilon_j - \varepsilon_k} \cdot v^+.$$

First, suppose $i = j-1$. We compute:

$$\begin{aligned} F_i E_i E_{\varepsilon_{i+1} - \varepsilon_k} \cdot v^+ &= -F_i E_{\varepsilon_i - \varepsilon_k} \cdot v^+ + q^{-1} F_i E_{\varepsilon_{i+1} - \varepsilon_k} E_i \cdot v^+ \\ &= -\kappa_{i+1, i, k} E_{\varepsilon_{i+1} - \varepsilon_k} \cdot v^+ + q^{-1} E_{\varepsilon_{i+1} - \varepsilon_k} F_i E_i \cdot v^+ \\ &= \mu_{i+1} [\mu_i; 1] E_{\varepsilon_{i+1} - \varepsilon_k} \cdot v^+ + q^{-1} [\mu_i; 1][\mu_{i+1}; 0] E_{\varepsilon_{i+1} - \varepsilon_k} \cdot v^+ \\ &= [\mu_i; 1][\mu_{i+1}; 1] E_{\varepsilon_{i+1} - \varepsilon_k} \cdot v^+ \\ &= [\bar{K}_i; 1][\bar{K}_{i+1}; 0] E_{\varepsilon_{i+1} - \varepsilon_k} \cdot v^+, \end{aligned}$$

and similarly if $i = k$. If $i = k - 1$ and $i > j$ then we compute:

$$\begin{aligned}
F_i E_i E_{\varepsilon_j - \varepsilon_{i+1}} \cdot v^+ &= q F_i E_{\varepsilon_j - \varepsilon_{i+1}} E_i \cdot v^+ \\
&= q^2 K_{j,i+1}^{-1} E_{-\varepsilon_i + \varepsilon_j} E_i \cdot v^+ + q E_{\varepsilon_j - \varepsilon_{i+1}} F_i E_i \cdot v^+ \\
&= \mu_j^{-1} \mu_{i+1} \kappa_{j,i,i+1} E_{\varepsilon_j - \varepsilon_{i+1}} \cdot v^+ + q[\mu_i; 1][\mu_{i+1}; 0] E_{\varepsilon_j - \varepsilon_{i+1}} \cdot v^+ \\
&= -\mu_{i+1}[\mu_i; 1] E_{\varepsilon_j - \varepsilon_{i+1}} \cdot v^+ + q[\mu_i; 1][\mu_{i+1}; 0] E_{\varepsilon_j - \varepsilon_{i+1}} \cdot v^+ \\
&= [\mu_i; 1][\mu_{i+1}; -1] E_{\varepsilon_j - \varepsilon_{i+1}} \cdot v^+ \\
&= [\bar{K}_i; 1][\bar{K}_{i+1}; 0] E_{\varepsilon_j - \varepsilon_{i+1}} \cdot v^+,
\end{aligned}$$

and similarly if $i = j$ and $i < k$. If $i = j$ and $i = k - 1$ it is easy to see that $F_i E_i E_i \cdot v^+ = [\bar{K}_i; 1][\bar{K}_{i+1}; 0] E_i \cdot v^+$. Since V is irreducible, it is generated by v^+ , hence it is equal to $U_q(\mathfrak{p}) \cdot v^+$. The result follows by induction on the degree of monomials in $U_q(\mathfrak{g})$. \square

3. CONSTRUCTION OF IRREDUCIBLE COMPLETELY POINTED MODULES

In this section we find a quantum version of the construction in [1] of irreducible completely pointed weight \mathfrak{gl}_{n+1} -modules. Then we show that any irreducible completely pointed $U_q(\mathfrak{gl}_{n+1})$ -module occurs in this way.

As in [6, Theorem 3.2], one checks that there is an \mathbb{F} -algebra homomorphism

$$\begin{aligned}
\pi : U_q(\mathfrak{gl}_{n+1}) &\longrightarrow A_{n+1}^q \\
E_i &\longmapsto x_i y_{i+1}, \\
F_i &\longmapsto x_{i+1} y_i, \\
\bar{K}_i &\longmapsto \omega_i.
\end{aligned} \tag{3.1}$$

where A_{n+1}^q is the *quantized Weyl algebra*, defined as the associative unital \mathbb{F} -algebra with generators $\omega_i, \omega_i^{-1}, x_i, y_i$, $i \in \{1, 2, \dots, n+1\}$ and defining relations

$$\omega_i \omega_j = \omega_j \omega_i, \tag{3.2a}$$

$$\omega_i \omega_i^{-1} = \omega_i^{-1} \omega_i = 1, \tag{3.2b}$$

$$\omega_i x_j \omega_i^{-1} = q^{\delta_{ij}} x_j, \tag{3.2c}$$

$$\omega_i y_j \omega_i^{-1} = q^{-\delta_{ij}} y_j, \tag{3.2d}$$

$$y_i x_j = x_j y_i, \quad i \neq j, \tag{3.2e}$$

$$y_i x_i - q^{-1} x_i y_i = \omega_i, \tag{3.2f}$$

$$y_i x_i - q x_i y_i = \omega_i^{-1} \tag{3.2g}$$

where $i, j \in \{1, 2, \dots, n+1\}$. The last two relations are equivalent to the two relations

$$y_i x_i = \frac{q\omega_i - (q\omega_i)^{-1}}{q - q^{-1}}, \quad x_i y_i = \frac{\omega_i - \omega_i^{-1}}{q - q^{-1}} \tag{3.2h}$$

Thus, A_{n+1}^q is isomorphic to the rank n generalized Weyl algebra $R(\sigma, t)$ where $R = \mathbb{F}[\omega_1^{\pm 1}, \dots, \omega_{n+1}^{\pm 1}]$, $\sigma_j(\omega_i) = q^{-\delta_{ij}}\omega_i$, $t_i = \frac{q\omega_i - (q\omega_i)^{-1}}{q - q^{-1}}$ for $i \in \{1, 2, \dots, n+1\}$.

The central element $I_{n+1} = \bar{K}_1 \cdots \bar{K}_{n+1}$ of $U_q(\mathfrak{gl}_{n+1})$ is mapped by π to the element $\mathbb{E}_q := \omega_1 \omega_2 \cdots \omega_{n+1}$. \mathbb{E}_q should be thought of as $q^{\sum_i x_i \partial_i}$: a q -analogue of the Euler operator.

Lemma 3.1. *The following identities hold.*

$$\mathbb{E}_q x_i \mathbb{E}_q^{-1} = q x_i, \quad \mathbb{E}_q y_i \mathbb{E}_q^{-1} = q^{-1} y_i, \quad \mathbb{E}_q \omega_i \mathbb{E}_q^{-1} = \omega_i, \quad i \in \{1, 2, \dots, n+1\}, \quad (3.3)$$

$$A_{n+1}^q = \bigoplus_{m \in \mathbb{Z}} A_{n+1}^q[m], \quad A_{n+1}^q[m] = \{a \in A_{n+1}^q \mid \mathbb{E}_q a \mathbb{E}_q^{-1} = q^m a\}, \quad (3.4)$$

$$A_{n+1}^q[m_1] \cdot A_{n+1}^q[m_2] \subseteq A_{n+1}^q[m_1 + m_2], \quad (3.5)$$

$$\pi(U_q(\mathfrak{gl}_{n+1})) = A_{n+1}^q[0] = C_{A_{n+1}^q}(\mathbb{E}_q). \quad (3.6)$$

where $C_{A_{n+1}^q}(\mathbb{E}_q)$ denotes the centralizer of \mathbb{E}_q in A_{n+1}^q .

Proof. The identities (3.3) follow directly from the commutation relations (3.2) in A_{n+1}^q . Identities (3.4) and (3.5) follow from (3.3) and that A_{n+1}^q is generated by x_i, y_i and ω_i . The second equality of (3.6) is trivial. By definition, (3.1), of π it is clear that $\pi(E_i), \pi(F_i), \pi(\bar{K}_i) \in A_{n+1}^q[0]$ for all $i \in \{1, \dots, n\}$. Since $U_q(\mathfrak{gl}_{n+1})$ is generated by the set $\{E_i, F_i, \bar{K}_i\}_{i=1}^{n+1}$, it follows that $\pi(U_q(\mathfrak{gl}_{n+1})) \subseteq A_{n+1}^q[0]$. It remains to prove that $A_{n+1}^q[0] \subseteq \pi(U_q(\mathfrak{gl}_{n+1}))$. First observe that $A_{n+1}^q[0]$ is invariant under left multiplication by elements from $R = \mathbb{F}[\omega_i^{\pm 1} \mid i = 1, \dots, n+1]$. Since A_{n+1}^q is a generalized Weyl algebra, it follows that $A_{n+1}^q[0]$ is generated as a left R -module by all monomials

$$a = x_1^{k_1} x_2^{k_2} \cdots x_{n+1}^{k_{n+1}} y_1^{l_1} y_2^{l_2} \cdots y_{n+1}^{l_{n+1}}$$

where $k, l \in (\mathbb{Z}_{\geq 0})^{n+1}$ are such that $\sum_i k_i = \sum_i l_i$ and $k_i \cdot l_i = 0$ for all $i \in \{1, 2, \dots, n+1\}$. Since any such monomial a is a product of elements of the form $x_i y_j$, where $i \neq j$, it suffices to show that $x_i y_j$ lies in the image of π for any $i \neq j$. We prove by induction on j that $x_i y_j \in \pi(U_q(\mathfrak{gl}_{n+1}))$ whenever $i < j$. If $j = i+1$, then $x_i y_{i+1} = \pi(E_i)$. If $j > i+1$, note that by (3.2f),

$$x_i y_j = \omega_{j-1}^{-1} [x_i y_{j-1}, \pi(E_{j-1})]_q \quad (3.7)$$

(recalling that $[a, b]_u := ab - uba$), which by the induction hypothesis lies in the image of π . Similarly one can use $\pi(F_i)$ to prove that $x_i y_j \in \pi(U_q(\mathfrak{gl}_{n+1}))$ if $i > j$. This finishes the proof of (3.6). \square

Lemma 3.2. *Let V be an irreducible A_{n+1}^q weight module and $\mathfrak{m} \in \text{Specm}(R)$ with $V_{\mathfrak{m}} \neq 0$. Then $\dim_{R/\mathfrak{m}} V_{\mathfrak{m}} = 1$. If in addition $\dim_{\mathbb{F}} R/\mathfrak{m} = 1$, then V is completely pointed.*

Proof. Let $A = A_{n+1}^q$. Since A is a generalized Weyl algebra and V is an irreducible weight A -module, each weight space $V_{\mathbf{m}}$ is an irreducible $C(\mathbf{m})$ -module, where $C(\mathbf{m}) = \bigoplus_{g \in \text{Stab}_{\mathbb{Z}^{n+1}}(\mathbf{m})} A_g$ is the cyclic subalgebra of A with respect to \mathbf{m} , (see e.g. [11, Prop. 7.1]). Since q is not a root of unity, the action of \mathbb{Z}^{n+1} on $\text{Specm}(R)$ is faithful, and therefore $C(\mathbf{m}) = R$, which is commutative. This implies that $\dim_{R/\mathbf{m}} V_{\mathbf{m}} \leq 1$. The second claim follows the fact that the support of an indecomposable weight module over a generalized Weyl algebra is invariant under the automorphisms $\sigma_1, \dots, \sigma_n$ and that $\dim_{\mathbb{F}} R/\mathbf{m} = \dim_{\mathbb{F}} R/\tau(\mathbf{m})$ for any \mathbb{F} -algebra automorphism τ of R . \square

Let $\text{Specm}^1(R)$ denote the set of all maximal ideals \mathbf{m} of R such that R/\mathbf{m} is one-dimensional over \mathbb{F} . Thus $\mathbf{m} = (\omega_1 - \mu_1, \dots, \omega_{n+1} - \mu_{n+1})$, where $(\mu_1, \dots, \mu_{n+1}) \in (\mathbb{F}^\times)^{n+1}$.

Theorem II A. *Let W be an irreducible completely pointed A_n^q -module. Let π^*W be the $U_q(\mathfrak{gl}_{n+1})$ -module, given as the π -pullback of W , where π is the map (3.1). Then π^*W is completely reducible, and each irreducible submodule is completely pointed, and occurs with multiplicity one.*

Proof. Since $\mathbb{E}_q \in R$, the A_{n+1}^q -module W decomposes in particular into eigenspaces with respect to \mathbb{E}_q . Due to the commutation relations in A_{n+1}^q , the ratio of any two eigenvalues is a power of q . That is, there exists a non-zero $\xi \in \mathbb{F}$ such that

$$W = \bigoplus_{m \in \mathbb{Z}} W[m], \quad W[m] = \{w \in W \mid \mathbb{E}_q w = \xi q^m w\} \quad (3.8)$$

Each $W[m]$ is a direct sum of certain R -weight spaces of W . More precisely, for each $m \in \mathbb{F}$ we have

$$W[m] = \bigoplus_{\substack{\mathbf{m} \in \text{Specm}^1(R) \\ \mathbb{E}_q - q^m \xi \in \mathbf{m}}} W_{\mathbf{m}}.$$

By Lemma 3.1, each subspace $W[m]$ is an $U_q(\mathfrak{gl}_{n+1})$ -submodule of W . Since $\pi(\bar{K}_i) = \omega_i$ for each $i \in \{1, \dots, n+1\}$, and W is completely pointed as an A_{n+1}^q -module, it follows that each $W[m]$, $m \in \mathbb{Z}$ is a completely pointed $U_q(\mathfrak{gl}_{n+1})$ -module. It remains to prove that for each $m \in \mathbb{Z}$, the $U_q(\mathfrak{gl}_{n+1})$ -module $W[m]$ is either zero or irreducible. By (3.6), proving that $W[m]$ is irreducible as an $U_q(\mathfrak{gl}_{n+1})$ -module is the same thing as proving that $W[m]$ is irreducible as an $A_{n+1}^q[0]$ -submodule of W .

Suppose $W[m] \neq \{0\}$ and let w_0 and w_1 be any two non-zero weight vectors of $W[m]$ of weights \mathbf{m}_0 and \mathbf{m}_1 respectively. A_{n+1} is generated as a left R -module by monomials of the form

$$a = x_1^{k_1} x_2^{k_2} \cdots x_{n+1}^{k_{n+1}} \cdot y_1^{l_1} y_2^{l_2} \cdots y_{n+1}^{l_{n+1}},$$

where $k, l \in (\mathbb{Z}_{\geq 0})^{n+1}$ and $k_i l_i = 0$ for each i . Moreover, there is at most one such monomial a such that $(aW_{\mathbf{m}_0}) \cap W_{\mathbf{m}_1} \neq \{0\}$. Since W is irreducible as an

A_{n+1} -module, there exists $r \in R$ and a single monomial a such that $raw_0 = w_1$. Since $w_0, w_1 \in W[m]$, this forces $\sum_i k_i = \sum_i l_i$, which implies that $a \in A_{n+1}^q[0]$. This proves that $W[m]$ is irreducible as an $A_{n+1}^q[0]$ -module. \square

The cyclic algebra of $U_q(\mathfrak{g})$ — $C(U_q(\mathfrak{g}))$ —is defined to be the subalgebra of all elements commuting with $K_i^{\pm 1}, i \in \{1, 2, \dots, n+1\}$.

Lemma 3.3. *Let V be an irreducible, infinite dimensional, completely pointed $U_q(\mathfrak{sl}_{n+1})$ -module. Then $\ker \pi \subseteq \text{Ann}_{U_q(\mathfrak{sl}_{n+1})} V$, where π is the map (3.1).*

Proof. Let $x \in C(U_q(\mathfrak{sl}_{n+1}))$. Write $x = \sum K F_i E_j$ where $K \in \mathbb{F}[K_1^{\pm 1}, K_2^{\pm 1}, \dots, K_n^{\pm 1}]$, and the sequences $\mathbf{i} = (i_1, i_2, \dots, i_l), \mathbf{j} = (j_1, j_2, \dots, j_l)$ are such that \mathbf{i} is a permutation of \mathbf{j} . We have:

$$\begin{aligned} \pi(x) &= \sum \pi(K) \pi(F_{\mathbf{i}}) \pi(E_{\mathbf{j}}) \\ &= \sum \pi(K) \prod_{r=1}^l (x_{i_r+1} y_{i_r}) \prod_{r=1}^l (x_{j_r} y_{j_r+1}) \\ &= \sum \pi(K) \prod_{r=1}^l ([\omega_{i_r}; s_r - s'_r][\omega_{j_r+1}; t_r - t'_r]) \end{aligned}$$

where s_r (resp. s'_r) denotes the number of times the element i_r (resp. $i_r - 1$) appears in the sequence $(j_1, j_2, \dots, j_l) \setminus (i_{r+1}, i_{r+2}, \dots, i_l)$ and t_r (resp. t'_r) denotes the number of times $j_r + 1$ (resp. j_r) appears in the sequence $(j_{r+1}, j_{r+2}, \dots, j_l)$. We prove this by induction on l . If $l = 1$ then we have $s_1 = 1, s'_1 = 0, t_1 = 0, t'_1 = 0$ and compute:

$$\begin{aligned} x_{i_1+1} y_{i_1} x_{i_1} y_{i_1+1} &= x_{i_1+1} \frac{(q\omega_{i_1}) - (q\omega_{i_1})^{-1}}{q - q^{-1}} y_{i_1+1} \\ &= [\omega_{i_1}; 1][\omega_{i_1+1}; 0] \end{aligned}$$

For $l > 1$ observe that $x_{i_l+1} y_{i_l}$ commutes with $x_{j_k} y_{j_k+1}$ if and only if $j_k \neq i_l$. Let $k \in \{1, 2, \dots, l\}$ be the minimum such that $i_l = j_k$ (we know such a k exists since \mathbf{j} is a permutation of \mathbf{i}). We have:

$$\begin{aligned} &\left(\prod_{r=1}^l x_{i_r+1} y_{i_r} \right) \left(\prod_{r=1}^l x_{j_r} y_{j_r+1} \right) \\ &= \left(\prod_{r=1}^{l-1} x_{i_r+1} y_{i_r} \right) \left(\prod_{r=1}^{k-1} x_{j_r} y_{j_r+1} \right) (x_{i_l+1} y_{i_l}) (x_{j_k} y_{j_k+1}) \left(\prod_{r=k+1}^l x_{j_r} y_{j_r+1} \right) \\ &= \left(\prod_{r=1}^{l-1} x_{i_r+1} y_{i_r} \right) \left(\prod_{r=1}^{k-1} x_{j_r} y_{j_r+1} \right) t_{i_l} \sigma_{j_k+1}(t_{j_k+1}) \left(\prod_{r=k+1}^l x_{j_r} y_{j_r+1} \right) \end{aligned}$$

$$= \left(\prod_{r=1}^{l-1} x_{i_r+1} y_{i_r} \right) \left(\prod_{k \neq r=1}^l x_{j_r} y_{j_{r+1}} \right) [\omega_{i_l}; s_l - s'_l][\omega_{j_k+1}; t_k - t'_k]$$

and apply induction to obtain the desired result.

Now, let V be an infinite-dimensional irreducible completely-pointed $U_q(\mathfrak{sl}_{n+1})$ -module. Then, using Theorem I, we extend the $U_q(\mathfrak{sl}_{n+1})$ action on V to a $U_q(\mathfrak{gl}_{n+1})$ action that satisfies $F_i E_i \cdot v = [\bar{K}_i; 1][\bar{K}_{i+1}; 0] \cdot v$ for all weight vectors $v \in V$. By a similar computation we see:

$$x \cdot v = \sum_{r=1}^l K \prod_{r=1}^l ([\bar{K}_{i_r}; s_r - s'_r][\bar{K}_{j_{r+1}}; t_r - t'_r]) \cdot v$$

where s_r, s'_r, t_r , and t'_r are as before. Therefore, $\pi(x)$ is a Laurent polynomial in the ω_i and x acts on v by the same Laurent polynomial evaluated at $\omega_i = \mu_i$. If $x \in \ker(\pi)$ then this Laurent polynomial must be identically 0, hence $x \cdot v = 0$. Since V is irreducible, v is a cyclic vector, hence $x \in \text{Ann}_{U_q(\mathfrak{sl}_{n+1})}(V)$.

Next we prove $\ker(\pi) \subseteq \text{Ann}_{U_q(\mathfrak{sl}_{n+1})}(V)$. Let $x \in \ker(\pi)$. Without loss of generality we can assume x is homogeneous with respect to the root lattice grading: $x = x_\beta$ for some $\beta \in Q$ and $K_i x K_i^{-1} = q^{\langle \beta, \alpha_i \rangle} x$ for $i = 1, 2, \dots, n$. Assume for the sake of contradiction that there exists an irreducible completely pointed weight module V for which $x \cdot V \neq 0$. Then there exists a weight vector $v \in V$ such that $w = x \cdot v \neq 0$. Since x is homogeneous, w is also a weight vector. Since V is irreducible, there exists some homogeneous element $y \in U_q(\mathfrak{sl}_{n+1})$ of degree $-\beta$ such that $y \cdot w = v$. Then yx has degree zero, and thus belongs to the centralizer $C(U_q(\mathfrak{sl}_{n+1}))$ of K_1, \dots, K_n . Also, yx belongs to $\ker(\pi)$ since it is an ideal in U_q . So $yx \in C(U_q(\mathfrak{sl}_{n+1})) \cap \ker(\pi)$ which by the previous paragraph implies that $(yx) \cdot V = \{0\}$, which contradicts the fact that $(yx) \cdot v_\lambda = v_\lambda \neq 0$. \square

Theorem II B. *Any infinite-dimensional irreducible completely pointed $U_q(\mathfrak{sl}_{n+1})$ is isomorphic to a direct summand of π^*W for some irreducible completely pointed A_{n+1}^q -module W .*

Proof. Let V be an irreducible completely pointed $U_q(\mathfrak{sl}_{n+1})$ -module, where we extend the action to make V become a $U_q = U_q(\mathfrak{gl}_{n+1})$ -module as in Theorem I. Consider the A_{n+1}^q -module

$$\widetilde{W} = A_{n+1}^q \otimes_{U_q} V$$

where A_{n+1}^q is regarded as a right U_q -module via the homomorphism $\pi : U_q \rightarrow A_{n+1}^q$ defined in (3.1). Since $A_{n+1}^q = \bigoplus_{m \in \mathbb{Z}} A_{n+1}^q[m]$ as right $A_{n+1}^q[0]$ -modules, we have

$$\widetilde{W} = \bigoplus_{m \in \mathbb{Z}} \widetilde{W}[m], \quad \widetilde{W}[m] = A_n^q[m] \otimes_{U_q} V. \quad (3.9)$$

Since V is completely pointed we have $V = \bigoplus_{\lambda \in (U_q^0)^*} V_\lambda$, $V_\lambda = \{v \in V \mid kv = \lambda(k)v, \forall k \in U_q^0\}$, where $(U_q^0)^*$ is the set of characters of $U_q^0 = \mathbb{F}[K_1^{\pm 1}, \dots, K_{n+1}^{\pm 1}]$.

Since $I_{n+1} := \bar{K}_1 \cdots \bar{K}_{n+1}$ is central in U_q and V is irreducible, it follows that I_{n+1} acts by some scalar $\xi \in \mathbb{F} \setminus \{0\}$ on V . Thus $\mathbb{E}_q a_m \otimes_{U_q} v_\lambda = q^m a_m \mathbb{E}_q \otimes_{U_q} v_\lambda = q^m a_m \otimes_{U_q} I_n v_\lambda = \xi q^m a_m \otimes_{U_q} v_\lambda$ for $m \in \mathbb{Z}$, $a_m \in A_{n+1}^q[m]$, $\lambda \in (U_q^0)^*$ and $v_\lambda \in V_\lambda$. Thus

$$A_{n+1}^q[m] \otimes V = \{w \in \widetilde{W} \mid \mathbb{E}_q w = \xi q^m w\}. \quad (3.10)$$

This turns \widetilde{W} into a \mathbb{Z} -graded A_{n+1}^q -module: $A_{n+1}^q[m_1] \widetilde{W}[m_2] \subseteq \widetilde{W}[m_1 + m_2]$ for all $m_1, m_2 \in \mathbb{Z}$. In particular $\widetilde{W}[0]$ is left $A_{n+1}^q[0]$ -submodule of \widetilde{W} , and can thus be regarded as a U_q -module via the map (3.1). By (3.6) and Lemma 3.3, there is a linear map

$$\begin{aligned} A_{n+1}^q[0] \otimes_{\mathbb{F}} V &\longrightarrow V, \\ \pi(a) \otimes v &\longmapsto av. \end{aligned} \quad (3.11)$$

It is balanced with respect to the right and left U_q -actions and hence induces a homomorphism $\widetilde{W}[0] \rightarrow V$, which is an isomorphism of U_q -modules with inverse $v \mapsto 1 \otimes_{U_q} v$. Let N be the sum of all \mathbb{Z} -graded submodules S of \widetilde{W} such that $S \cap \widetilde{W}[0] = \{0\}$. Then N is the unique maximal \mathbb{Z} -graded submodule of \widetilde{W} with $N \cap \widetilde{W}[0] = \{0\}$. Define

$$W = \widetilde{W}/N. \quad (3.12)$$

Since N is graded, so is W , and (3.11) and $N \cap \widetilde{W}[0] = \{0\}$ imply that V is isomorphic to $W[0]$, which is a direct summand of W as a left $A_{n+1}^q[0]$ -module, hence as a left U_q -module via (3.1).

We show that W is an irreducible A_{n+1}^q -module. If M is a nonzero submodule of W , then the inverse image, \widetilde{M} , of M under the canonical projection $\widetilde{W} \rightarrow W$ is a graded nonzero module containing N and thus must have nonzero intersection with $\widetilde{W}[0]$ by maximality of N . Since V is an irreducible U_q -module, $\widetilde{W}[0]$ is an irreducible $A_n^q[0]$ -module. Therefore

$$\widetilde{M} \supseteq A_{n+1}^q[0] \widetilde{M} \supseteq A_{n+1}^q[0] \widetilde{W}[0] = \widetilde{W}[0].$$

But \widetilde{W} is generated as a left A_{n+1}^q -module by $\widetilde{W}[0]$ which implies that $\widetilde{M} = \widetilde{W}$ and thus $M = W$.

It remains to be proved that W is completely pointed. Let λ be a character of V such that $V_\lambda \neq \{0\}$ and let $v_\lambda \in V_\lambda \setminus \{0\}$. Define $\mathbf{m} = (\omega_1 - \lambda(\bar{K}_1), \dots, \omega_{n+1} - \lambda(\bar{K}_{n+1})) \in \text{Specm}^1(R)$. Then the vector $(1 \otimes_{U_q} v_\lambda) + N$ is a nonzero R -weight vector of weight \mathbf{m} , so by Lemma 3.2, W is a completely pointed A_{n+1}^q -module. \square

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